Liquidation in the Face of Adversity: Stealth Versus Sunshine Trading

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Abstract

We consider a multi-player situation in an illiquid market in which one player tries to liquidate a large portfolio in a short time span, while some competitors know of the seller's intention and try to make a profit by trading in this market over a longer time horizon. We show that the liquidity characteristics, the number of competitors and their trading time horizons determine the optimal strategy for the competitors: they either provide liquidity to the seller, or they prey by simultaneous selling. Depending on the expected competitor behavior, it might be sensible for the seller to pre-announce a trading intention ("sunshine trading") or to keep it secret ("stealth trading").

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Introduction

A variety of circumstances such as a margin call or a stop-loss strategy in combination with a large price drop can force a market participant to liquidate a large asset position urgently. Such a swift liquidation may result in a significant impact on the asset price. Hence, intuitively it seems to be crucial to prevent information leakage while executing the trade, for informed market participants (the "competitors") could otherwise try to earn a profit by predatory trading: they can sell in parallel with the seller and cover their short positions later at a lower price. Probably the most widely known example of such a situation is the alleged predation on the hedge fund LTCM[1]. Surprisingly, however, some sellers do not follow a secretive "stealth trading" strategy but rather practice "sunshine trading", which consists in pre-announcing the trade to competitors so as to attract liquidity[2].

Our goal in this paper is to analyze a model of a competitive trading environment in order to explain the tradeoff that leads the seller to choose between stealth and sunshine execution and the competitors to choose between predation and liquidity provision. We argue that these choices are driven by the relations between the different liquidity parameters of the market, the number of competitors and the time constraints for seller and competitors. In particular, we will show that completely different behavioral patterns may emerge as optimal for the same set of agents when they are trading in markets of different liquidity types. Since our model market allows for anonymous trading possibilities and will turn out as being semi-strong efficient, our results are applicable to a wide variety of real-world markets including most equity exchange markets.

To fully acknowledge the roles of the different liquidity parameters and of the number of competitors of the seller, we need to relax all exogenous trading constraints in our model. In particular, we do not require that competitors face the same time constraint as the seller. This assumption is reasonable as sellers typically must achieve a trading target in a fixed and relatively short time horizon—e.g., a margin call has to be covered by the end of the day—while competitors often may afford to maintain a long or a short position for a number of days. In order to capture the structure of this situation, we consider a two stage model of an illiquid market. In the first stage, the seller as well as the competitors trade; in the second stage, only the competitors trade and unwind the asset positions they acquired during the first stage. Liquidity effects are incorporated into our market model by applying a permanent as well as a temporary impact as in the market model proposed by Almgren and Chriss (2001) and used by Carlin, Lobo, and Viswanathan (2007). For the sake of simplicity, throughout this paper we focus on the liquidation of a long position of assets; equivalent statements hold for the liquidation of a short position.

In our analysis of the optimal agent behavior in this model, we first assume that all strategic agents know the seller's liquidation intentions. We derive a Nash equilibrium of optimal trading strategies for the seller and the competitors, and we show that, in equilibrium, the competitors' optimal strategy depends heavily on the liquidity type of the market. We identify two distinct types of illiquid markets: First, if the temporary price impact dominates the permanent impact then prices show a high resilience after a large transaction. The price in such "elastic" markets behaves similar to a rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers. Such market conditions can occur when it is difficult to find counterparties for a specific deal within a short time. In such a market, the optimal strategy for the competitors is to cooperate with the seller: they should buy some of the seller's assets and sell them at a later point in time. On the other hand, if the permanent price impact of a trade outweighs the temporary impact, then large transactions have a long-lasting influence. In such "plastic" markets, the trading pressure exerts a "plastic deformation" on the market price. Such a situation can arise when a large supply or demand of the asset is interpreted as the revelation of new information on the fundamentals of the asset. Under these conditions the optimal behavior of the competitors is the opposite: they should sell in parallel to the seller and buy back at a later point in time (predatory

[1]See, e.g., Lowenstein (2001), Jorion (2000) and Cai (2003)).

[2]See, e.g., Harris (1997) and Dia and Pouget (2006). A similar phenomenon occurs in the sometimes widespread distribution of so-called "indications of interest" in which brokers announce tentative conditions for certain liquidity trades.

trading). In this case, the price is pushed far down during the first stage and recovers during the second stage, resulting in price overshooting. The latter effect disappears as the number of competitors increases; for a large number of competitors, the market price incorporates the seller's intentions almost instantly and exhibits little drift thereafter. This effect indicates that our model market fulfills the semi-strong form of the efficient markets hypothesis.

Through sunshine trading, the seller can increase the number of competitors. We find that in elastic markets, the seller achieves a higher return when competitors are participating than when she is selling by herself. Therefore, sunshine trading appears to be sensible in such a market. In a plastic market, the seller's return can be significantly reduced by competitors; however, as the number of competitors increases, the optimal strategies for the competitors change from predation to cooperation and the return for the seller increases back, sometimes even above the level of return obtained in the absence of competitors. Hence, if the seller has reason to believe that there is *some* leakage of information, it may be sensible to take the initiative of publicly announcing the impending trade so as to turn around the adverse situation of predation by few competitors into the beneficial situation of liquidity provision.

Although our approach is normative rather than descriptive, our model provides a number of empirically testable hypotheses for both seller and competitor behavior. In our model, sunshine trading is rational in elastic markets or when the trading horizon of the seller is comparatively short. We therefore suspect that sunshine trades and indications of interest are usually short-term and occur in markets with high temporary impact, while we conjecture that efforts to conceal trading intentions are particularly strong in plastic markets.

We predict that competitors in plastic markets pursue predatory trading if they know about selling intentions of other agents, while we expect them to provide liquidity in elastic markets. We are not yet aware of any systematic study of informed competitors reactions to trading under varying market liquidity. Such a study could be carried out, e.g., by analyzing the order flow after pre-announcement of a sale. In plastic markets, we expect to see an initial increase in additional sell orders. In elastic markets, we expect to see an increase in buy orders.

The analysis of distressed hedge funds lends anecdotal support to our hypothesis. During the LTCM crisis in 1998, several competitors allegedly engaged in front-running and predatory trading, while no individual investor was willing to acquire LTCM's positions and thus provide liquidity. According to our results, such a behavior is rational in plastic markets. The price evolution after the LTCM crisis indicates that its liquidation had a predominantly permanent effect[3], i.e., that the market was indeed plastic.

More recently, the hedge fund Amaranth experienced severe losses resulting in the need for urgent liquidation[4]. Contrary to LTCM, Amaranth quickly found a buyer for its portfolio[5]. In the Amaranth case, liquidity provision apparently appeared as the more profitable option for competitors compared to predatory trading. How can the differences between competitors' behavior in the LTCM and Amaranth cases be explained? In both cases very large market participants were in distress, promising large profit opportunities for competitors. However, Amaranth operated in the natural gas market, which behaved elastic during a previous hedge fund liquidation[6]. According to our model liquidity provision is indeed the most profitable behavior in such an elastic market.

The profitability of liquidity provision in elastic markets is confirmed by Coval and Stafford (2007),

[3] Lowenstein (2001, page 229) notes that: "(...) a year after the bailout [of LTCM], swap spreads remained (...) far higher than when Long-Term had entered the (...) trade."

[4] For a description of the Amaranth case, see Till (2006) and Chincarini (2007). Finger (2006) finds that "The events of September [2006] led to the greatest losses ever by a single hedge fund, close to twice the money lost by Long Term Capital Management."

[5] Till (2006) notes that "Amaranth sold its entire energy-trading portfolio to J.P. Morgan Chase and Citadel Investment Group on Wednesday, September 20th [2006]."

[6] Till (2006) observes that "There was a preview of the intense liquidation pressure on the Natural Gas curve on 8/2/06, the day before the [natural-gas-oriented] energy hedge fund, MotherRock, announced that they were shutting down. (...) A near-month calendar spread in Natural Gas experienced a 4.5 standard-deviation move intraday before the spread market normalized by the close of trading on 8/2/06."

who find that providing liquidity to open-ended mututal funds that suffer severe cash outflows promises average annual abnormal excess returns well over 10%. This supports our hypothesis since these profits are made on the temporary nature of the price impact. Interestingly, the impact of stock sales in markets that do not suffer from extreme cash outflows appears to be predominantly permanent, resulting in profitable predatory trading opportunities for insiders.

Our research builds on previous work in three research areas. The first area to which our work is connected is research on predatory trading. In previous studies, the size of the liquidation completely determines the optimal action of the competitors. In these models, predatory trading is always optimal for large liquidations. For small liquidations, predatory trading is always or never optimal, depending on the model at hand.

Brunnermeier and Pedersen (2005) suggest a model in which the total rate of trading as well as the asset positions of all traders face exogenous constraints. In their model, predation and price overshooting occur necessarily, irrespective of the market environment. As a side effect of the exogenous trading constraint, their model market is weakly inefficient: even if the number of informed competitors is large, the market price changes continuously in reaction to the trading of the seller and the competitors.

Carlin, Lobo, and Viswanathan (2007) propose a model in which competitors can engage in and refrain from predatory trading, however there is no room for liquidity provision. To explain abstinence from predatory trading, they assume that all market participants repeatedly execute large transactions in a fully transparent market; in such a repeated game, predation can be punished by applying a tit-for-tat strategy. In their model, competitors always refrain from predatory trading provided that the position of the seller is sufficiently small. Abstinence from predatory trading breaks down, however, if the sales volume becomes too big. Although their analysis of a one stage game is also at the foundation of our model, the two models diverge in their qualitative predictions: their model predicts that predatory trading is most widespread in elastic markets, while our model predicts the opposite. Furthermore, the mechanism establishing cooperation is fundamentally different; in particular, our model is applicable in transparent, but also in anonymous markets.

Attari, Mello, and Ruckes (2005) discuss trading strategies against a financially constraint arbitrageur. Price impact in their model market is entirely temporary, resulting in an elastic market with profitable liquidity provision. By clever exploitation of the arbitrageur's capital constraint, the competitors can profitably engage in predatory trading, but only for arbitrageurs with very large asset positions.

In a second line of research, the effects of sunshine trading are investigated. In a theoretical investigation, Admati and Pfleiderer (1991) propose a model in which sunshine trading is always increasing the seller's return as long as speculators do not face market entry costs. The underlying motives for sunshine trading in this model and in our model are very different: in the model of Admati and Pfleiderer (1991), sunshine traders can expect to obtain better trade conditions in the market since it is assumed that their actions are not based on private information. In our model, we do not assume that sunshine trades have a special motivation; instead, we show that sunshine trading under certain market conditions can raise the attention of competitors and attract them to provide liquidity. A different market perception of sunshine trades can easily be incorporated in our framework by applying different liquidity parameters for sunshine trades and for unannounced trades. Empirical evidence on the benefit of trade pre-announcements appears to be mixed (see, e.g., Harris (1997), Dia and Pouget (2006)), which is in line with our observation that the potential benefit of sunshine trading depends on the liquidity characteristics of the market.

The third line of research consists of empirical investigations and theoretical modeling of the market impact of large transactions. The empirical literature is extensive[7]. These empirical results, most notably the identification of temporary and permanent impact, have led to theoretical models of illiquid markets.

[7] See, e.g., Kraus and Stoll (1972), Holthausen, Leftwich, and Mayers (1987), Holthausen, Leftwich, and Mayers (1990), Barclay and Warner (1993), Chan and Lakonishok (1995), Biais, Hillion, and Spatt (1995), Kempf and Korn (1999), Chordia, Roll, and Subrahmanyam (2001), Chakravarty (2001), Lillo, Farmer, and Mantegna (2003), Mönch (2004), Almgren, Thum, Hauptmann, and Li (2005), Coval and Stafford (2007), Obizhaeva (2007), Large (2007).

One line of research focused on deriving the underlying mechanisms for these liquidity effects[8]. A second line takes the liquidity effects as exogeneously given and derives optimal trading strategies within such an idealized model market. We follow this second approach and apply the multi-player market model of Carlin, Lobo, and Viswanathan (2007). Several alternative single-player models have been proposed, see, e.g., Bertsimas and Lo (1998), Almgren and Chriss (2001), Almgren (2003), Butenko, Golodnikov, and Uryasev (2005), Obizhaeva and Wang (2006), Engle and Ferstenberg (2007), Alfonsi, Schied, and Schulz (2007), Frey (1997), Frey and Patie (2002), Bank and Baum (2004), Çetin, Jarrow, and Protter (2004), Çetin, Jarrow, Protter, and Warachka (2006), Jarrow and Protter (2007). The advantages and disadvantages of these models are still a topic of ongoing research.

The remainder of this paper is structured as follows. In Section I, we introduce the market model and specify the game theoretic optimization problem. As a preparation for the general two stage model, we review predation in a one stage model in Section II. In this model, the seller and the competitors face the same time constraint, i.e., the competitors do not have the opportunity to trade after the seller finished selling. In the main Section III, we turn to the more general two stage framework and derive our main results. After introducing the model in Subsection A, we identify the Nash equilibrium of optimal trading strategies in Subsection B and summarize the general properties in Subsection C. Thereafter, we investigate the qualitative properties of our model in three example markets in Section IV. Section V concludes. Appendix A contains additional propositions on the one stage model. All proofs of propositions are given in Appendix B.

I The market model

We start by describing the market dynamics and trade motives of market participants. The market consists of a risk-free asset and a risky asset. Trading takes place in continuous time. We assume that the risk-free asset does not generate interest. In this market we consider $n+1$ strategic players and a number of noise traders. The strategic players are aware of liquidity needs in the market and optimize their trading to profit from these needs. We assume that the number of strategic players $(n+1)$ is given a priori. During our analysis, we will perform comparative statics and discuss the incentives for each player to change the number of strategic players in the market.

We denote the time-dependent risky asset positions of the strategic players by $X_0(t)$, $X_1(t)$, ..., $X_n(t)$ and assume that they are differentiable in t. Each instantaneous order $\dot{X}_i(t)$ affects the market price in the form of a permanent impact and a temporary impact. Trades at time t are thus executed at the price

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t). \tag{1}$$

Here, $\tilde{P}(t)$ is a one-dimensional arithmetic Brownian motion without drift, starting at $\tilde{P}(0) = P_0$ and defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. This term reflects the price changes due to the random trades of noise traders. The second term on the right hand side represents the permanent price impact resulting from the change in total asset position of all strategic players. Its magnitude is determined by the parameter $\gamma > 0$. The third term reflects the temporary impact caused by the net trading speed of all strategic investors. Its magnitude is controlled by the parameter $\lambda > 0$. This price dynamics model was introduced by Carlin, Lobo, and Viswanathan (2007) and extends the previous single-player models of Bertsimas and Lo (1998), Almgren and Chriss (2001) and Almgren (2003).

In this market, the strategic players are facing the following optimization problem. Each player i knows all other players' initial asset positions $X_j(0)$ and their target asset positions $X_j(T)$ for some fixed

[8] See for example Kyle (1985), Glosten and Milgrom (1985), Easley and O'Hara (1987), Foster and Viswanathan (1996), Frey (1997), O'Hara (1998), Bondarenko (2001) and Biais, Glosten, and Spatt (2005).

point $T > 0$ in the future[9]. We assume that these trading targets are binding; players are not allowed to violate their targets. We furthermore assume that all agents are risk-neutral; they aim at maximizing their expected return by choosing an optimal trading strategy $X_i(t)$ given their boundary constraints on $X_i(0)$ and $X_i(T)$. In mathematical terms, each player is looking for a strategy that realizes the maximum

$$r_i := \max_{X_i} \mathbb{E}(\text{Return for player } i) = \max_{X_i} \mathbb{E}\left(\int_0^T (-\dot{X}_i(t))P(t)dt\right) \tag{2}$$

$$= \max_{X_i} \mathbb{E}\left(-\int_0^T \dot{X}_i(t)\left(\tilde{P}(t) + \gamma\sum_{j=0}^n (X_j(t) - X_j(0)) + \lambda\sum_{j=0}^n \dot{X}_j(t)\right)dt\right). \tag{3}$$

Although in principle the strategies X_i might be predictable, we limit our discussion to deterministic[10] strategies, where the function X_i does not depend on the stochastic price component $\tilde{P}(t)$. In such open-loop strategies, all players determine their trade schedules ex ante[11]. Hence,

$$r_i = \max_{X_i}\left(-\int_0^T \dot{X}_i(t)\left(P_0 + \gamma\sum_{j=0}^n (X_j(t) - X_j(0)) + \lambda\sum_{j=0}^n \dot{X}_j(t)\right)dt\right). \tag{4}$$

A set of strategies $(X_0, X_1, ..., X_n)$ satisfying Equation (4) for all $i = 0, 1, ..., n$ constitutes a Nash equilibrium; we call such a set of strategies *optimal*. We denote the returns corresponding to the equilibrium strategies by $R_i := r_i$. These are determined by the *expected* price

$$\bar{P}(t) := \mathbb{E}(P(t)) = P_0 + \gamma\sum_{i=0}^n (X_i(t) - X_i(0)) + \lambda\sum_{i=0}^n \dot{X}_i(t). \tag{5}$$

Whenever we refer to *price* or *return* in the following, we will always refer to the *expected price* $\bar{P}(t)$ and the *expected return* $-\int \dot{X}_i(t)\bar{P}(t)dt$ in equilibrium.

II The one stage model

In this section, we investigate the optimal strategies in a one stage framework: *all* players trade over the same time interval $[0, T_1]$. The results in this section will be used in the analysis of a two stage model in the following sections.

The one stage framework was introduced by Carlin, Lobo, and Viswanathan (2007). We repeat their result on optimal strategies in this setting:

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[9] For the purposes of this paper, we assume that all strategic players have perfect information. For imperfect information, we expect to obtain slightly changed dynamics (potentially including a "waiting game" as in Foster and Viswanathan (1996)), but expect the qualitative results on predatory trading and liquidity provision to remain unchanged.

[10] In the following, we will find a Nash equilibrium within the set of deterministic strategies. Since we assumed risk neutrality of all agents, these strategies also constitute a Nash equilibrium within the larger set of open-loop adaptive strategies. The primary purpose of our restriction to deterministic strategies is to reduce the mathematical complexity. Schied and Schöneborn (2008) showed that the unique optimal *adaptive* strategy for a single seller is *deterministic*. We thus believe that also in the multi-player setting all Nash equilibria in the set of adaptive strategies are actually deterministic and that therefore our analysis identified the unique equilibrium.

[11] The analysis of closed-loop strategies in which players can dynamically react to other players actions is mathematically more difficult. It is often not possible to derive closed-form solutions, on which we rely in the proof of Theorem 2. Carlin, Lobo, and Viswanathan (2007) show numerically that closed-loop solutions of the one stage model (see Section II) are similar to the open-loop solutions and do not exhibit any new qualitative features. Therefore, no major differences are expected in the two stage model introduced in Section III.

Parameter	Value
Asset position X_0	1
Initial price P_0	10
Duration T_1	1
Permanent impact sensitivity γ	3
Temporary impact sensitivity λ	1

Table I: Parameter values used for numerical computation of the figures in Section II.

Theorem 1 (Carlin, Lobo, and Viswanathan (2007))**.** *Assume that $n+1$ players are trading simultaneously in a time period $t \in [0, T_1]$. They start with asset positions $X_i(0)$ and need to achieve a target asset position $X_i(T_1)$. Furthermore, these players are risk-neutral and are aware of all other players' asset positions and trading targets. Then the unique optimal strategies for these $n+1$ players (in the sense of a Nash equilibrium) are given by:*

$$\dot{X}_i(t) = a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t} \tag{6}$$

with

$$a = \frac{n}{n+2}\frac{\gamma}{\lambda}\left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1} \frac{\sum_{i=0}^{n}(X_i(T_1) - X_i(0))}{n+1} \tag{7}$$

$$b_i = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)^{-1}\left(X_i(T_1) - X_i(0) - \frac{\sum_{j=0}^{n}(X_j(T_1) - X_j(0))}{n+1}\right). \tag{8}$$

Proof. See Carlin, Lobo, and Viswanathan (2007). □

The previous theorem assumes that $n \geq 1$, i.e., at least two strategic players are active in the market. It follows from the results in Almgren and Chriss (2001) and Almgren (2003) that for a single strategic trader the optimal trading strategy is a linear increase / decrease of the asset position.

For the rest of this section, we consider the following specific situation: One player (say player 0) wants to sell an asset position $X_0(0) = X_0$ in the time interval $[0, T_1)$, i.e. the target is given by $X_0(T_1) = 0$. All other players (i.e., players 1, 2, ..., n) do not want to change their initial and terminal asset positions (for simplicity, we assume that $X_i(0) = X_i(T_1) = 0$ for $i \neq 0$), but they want to exploit their knowledge of player 0's sales.

The result is *preying* of the n players on the first player (see Figure 1 and 2; see Table I for the parameter values used for the figures): while the first player is starting to sell off her asset position, the other players sell short the asset and realize a comparatively high price per share. Toward the end of the trading period, the price has been pushed down by the combined sales of both seller and competitors. While the seller liquidates the remaining part of her long position at a fairly low price, the other players can now close their short positions at a favorable price. Since the price has dropped, the preying players need to spend less on average for buying back than they received for initially selling short. In the following, we refer to player 0 as the "seller" and to the players 1, 2, ..., n as the "competitors".

In the one stage model considered so far, there is no room for cooperation; preying **always** occurs. The seller's return is further deteriorating as the number of competitors increases; preying becomes more competitive with more players being involved (see Figure 3). We will see in the next section that relaxing the exogenous time constraint on the positions of competitors can lead to a more differentiated behavior. It includes in particular the possibility of liquidity provision to the seller.

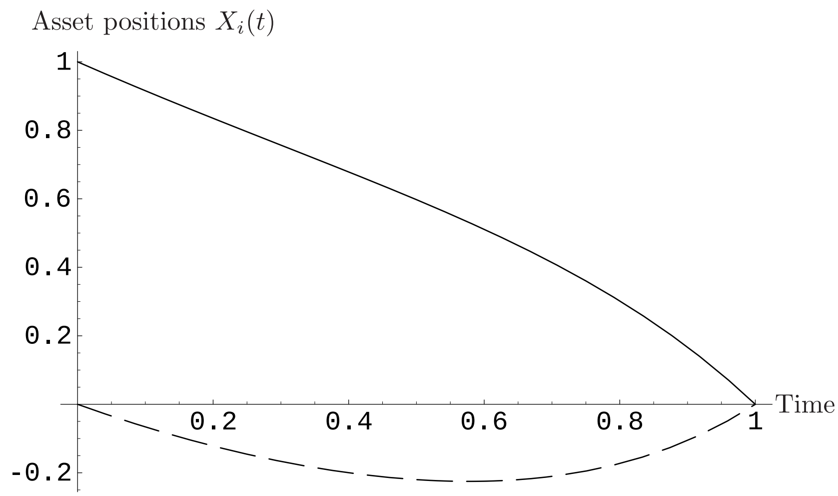

Figure 1: Asset positions $X_i(t)$ over time in the one-stage model. The solid line represents the seller, the dashed line the competitor ($n = 1$).

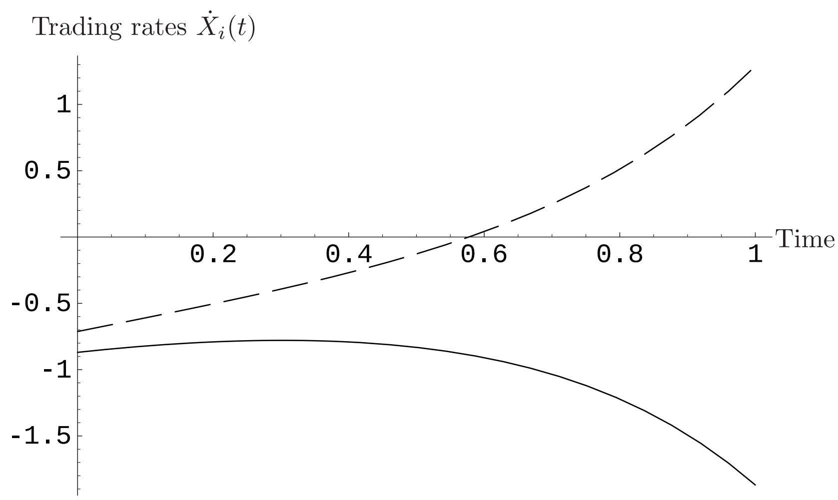

Figure 2: Trading rates $\dot{X}_i(t)$ over time in the one-stage model. The solid line represents the seller, the dashed line the competitor ($n = 1$).

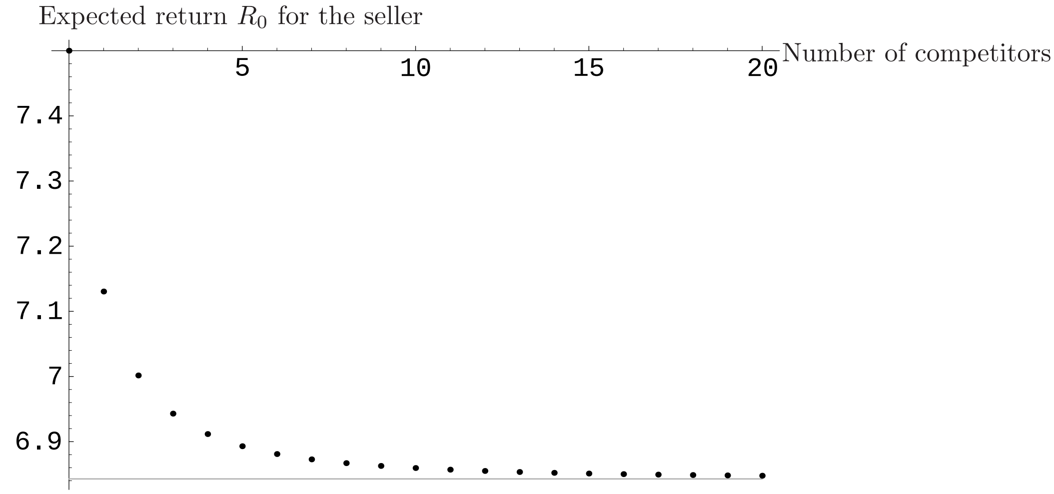

Figure 3: Expected cash return for the seller (player 0) from selling X_0 shares, depending on the number n of competitors in the one-stage model. The expected return in absence of competitors is 7.5 (intersection point of x- and y-axes). The grey line at the bottom corresponds to the limit $n \to \infty$.

III The two stage model

A The model

In the previous section, we have assumed that the seller and the competitors are limited to trade during the same time interval. As we have mentioned earlier, in reality the seller is often facing a stricter time constraint than the competitors do. While the seller usually needs to liquidate her asset position within a few hours, the competitors can often afford to close their positions at a later point in time. In the following, we therefore extend the one stage model considered so far to a two stage framework[12] and assume that:

- In stage 1, *all* players (the seller and the competitors) are trading.
- In stage 2, *only* the competitors are trading; the seller is not active.

The first stage runs from $t = 0$ to T_1, the second stage[13] from T_1 to T_2. The asset position of player i is denoted by $X_i(t)$ with $t \in [0, T_2]$. We require the strategies $X_i(t)$ to be differentiable within each stage, but they need not be differentiable at $t = T_1$.

[12]The framework can be extended further to a three stage model including a stage 0 in which only the competitors are allowed to trade. Such a setup can capture the effects of front-running, which produces different results in particular for price overshooting. We limit our analysis to the two stage model since in most practical cases, there is little room for front-running due to legal constraints or insufficient time (i.e., stage 0 is very short); see the introduction for examples.
As another alternative, the model can account for a different trading horizon for each competitor. This increases the mathematical complexity, but does not lead to qualitatively new phenomena within stage 1.

[13]In reality, the seller usually has to liquidate an asset position by the end of the trading day. In this case, the second stage begins at the open of the next trading day. Our framework can easily be extended to accommodate for this setting by having the second stage run from $\tilde{T}_1 > T_1$ to T_2. Since we assumed that the seller and the competitors are risk-neutral, this does not change any of the statements in this exposition; for notational simplicity, we therefore restrict ourselves to the case where the second stage starts immediately after the first stage.

The market prices are governed by

$$P(t) = \tilde{P}(t) + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t) \tag{9}$$

for $t \in [0, T_2] \backslash T_1$. Again, $\tilde{P}(t)$ is an arithmetic Brownian motion without drift, starting at $\tilde{P}(0) = P_0$. Since the $X_i(t)$ might be non-differentiable at $t = T_1$, the above formula might not be well-defined; we therefore set

$$P(T_1) = \lim_{t \searrow T_1} P(t), \qquad\qquad P(T_1-) := \lim_{t \nearrow T_1} P(t), \tag{10}$$

forcing the price to be right-continuous.

The seller (player 0) is assumed to liquidate an asset position $X_0 = X_0(0) > 0$ during stage 1: $X_0(t) = 0$ for all $t \in [T_1, T_2]$. We assume that the n competitors want to exploit their knowledge of the seller's intentions, but do not want to change their asset position permanently. We therefore require that the competitors have the same asset positions at the beginning of stage 1 and at the end of stage 2: $X_i(0) = X_i(T_2)$. For notational simplicity, we assume $X_i(0) = 0$. This does not limit the generality of our results, since the optimal trading speed $\dot{X}_i(t)$ of the competitors is independent of their initial asset position $X_i(0)$, as long as we require $X_i(0) = X_i(T_2)$. In particular, our results also hold in the case where competitors have different initial asset positions. All assumptions and notation introduced in Section I apply in our two stage model; in particular, we restrict our analysis to risk-neutral players[14] following deterministic strategies.

There are no a-priori restrictions on competitors' asset positions $X_i(T_1)$ at the end of stage 1. They can be positive, i.e., the competitors buy some of the seller's shares in stage 1 and thereby provide liquidity to the seller. Alternatively, they can be negative, i.e., the competitors sell parallel to the seller, driving the market price further down and preying on the seller. In the next section, we show that the occurrence of liquidity provision or predation depends on the market characteristics, in particular on the balance between temporary and permanent impact.

B Optimal strategies

We can now describe the optimal behavior of all $n+1$ strategic players in the two stage model introduced in the previous subsection. If the optimal asset positions $X_i(T_1)$ of the competitors at the end of stage 1 are known, the entire optimal strategies are determined by Theorem 1: In stage 1, $n + 1$ players are trading and the initial and final asset positions are known; in stage 2, n players are trading and again the initial and final asset positions are known. Therefore, we only need to derive the optimal asset positions[15] $X_i(T_1)$ for all competitors $i = 1, 2, ..., n$ (see Figure 4 for an illustration).

Theorem 2. *A unique open-loop Nash equilibrium exists in the two stage model defined in Subsection A. In this equilibrium, all competitors acquire the same asset position during stage 1. This asset position*

[14] Risk aversion can be incorporated in two different ways. The first is to regard the different execution time frame of the seller and the competitors as proxies of their risk aversion. This provides a simple model of a highly risk averse seller in a market environment with relatively risk-neutral competitors. Alternatively, risk aversion can explicitly be modeled by introducing utility functions for the seller and the competitors. This leads to the coexistence of liquidity provision and preying already in the one stage model introduced in Section II. The dynamics for a risk averse seller facing relatively risk-neutral competitors are qualitatively very similar to the two stage model presented here. A detailed discussion of the effects of risk aversion lie beyond the scope of this paper and are subject of ongoing research.

[15] Carlin, Lobo, and Viswanathan (2007) noted this for the single competitor case. They also conjectured that in a two stage model there will be price overshooting. As we will see in Section IV and Proposition 14, the source of this price overshooting is not necessarily the presence of strategic players. In fact, price overshooting is reduced by competitors in elastic markets.

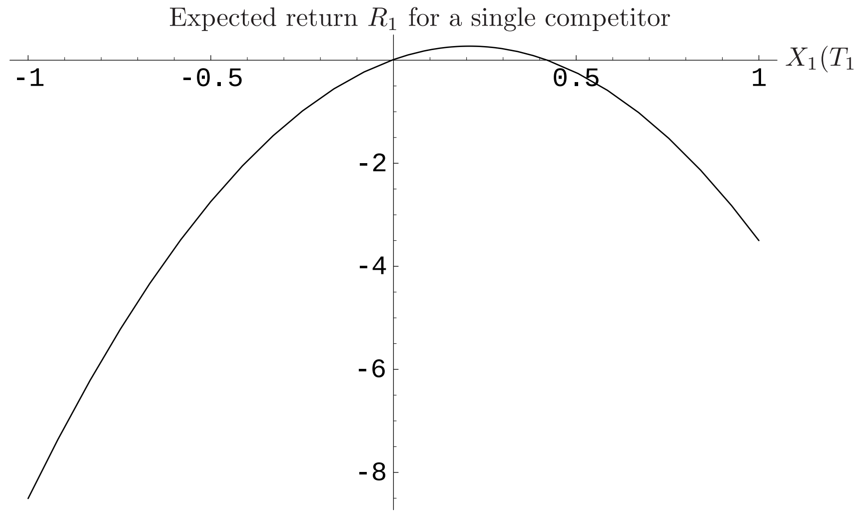

Figure 4: Expected return R_1 for a single competitor depending on her asset position $X_1(T_1)$. Optimal trading within stage 1 and stage 2 is assumed. Parameters are chosen as in the elastic market in Table II.

depends only on the parameters $\frac{\gamma T_1}{\lambda}$, $\frac{T_2}{T_1}$ *and* n*:*

$$X_i(T_1) = F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) X_0 \text{ for all } i \in 1, \ldots, n. \tag{11}$$

The function F *is given in closed form in the proof in Appendix B. For the special case* $n = 1$*, we obtain*

$$X_1(T_1) = -\frac{\left(-2 - e^{\frac{\gamma T_1}{3\lambda}} - e^{\frac{2\gamma T_1}{3\lambda}} + e^{\frac{\gamma T_1}{\lambda}}\right) \frac{\gamma}{\lambda}(T_2 - T_1)}{6\left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) + \left(2 + e^{\frac{\gamma T_1}{3\lambda}} + e^{\frac{2\gamma T_1}{3\lambda}} + 2e^{\frac{\gamma T_1}{\lambda}}\right) \frac{\gamma}{\lambda}(T_2 - T_1)} X_0. \tag{12}$$

Formulas 11 and 12 do not depend on γ and λ separately, but only on the fraction[16] $\frac{\gamma T_1}{\lambda} = \frac{\gamma}{\lambda/T_1}$, which can be interpreted as a normalized ratio of liquidity parameters. The permanent impact parameter γ has unit "dollars per share" and is independent of the time unit. The temporary impact parameter λ has unit "dollars per share per time unit" and thus depends on the time unit. The fraction λ/T_1 can be interpreted as the temporary impact parameter normalized to the length of the first stage.

It is not hard to see why the optimal asset position $X_i(T_1)$ does not depend on λ and γ individually, but only on their ratio γ/λ. For any given strategies for the seller and the competitors, doubling both γ and λ will double the liquidity loss of the seller and double the proceeds of each competitor. Since this linear scaling holds irrespective of the chosen strategy, we find that a Nash equilibrium with respect to the original parameters γ and λ is also a Nash equilibrium with respect to the parameter 2γ and

[16] Since the dependence of F on n is non-reciprocal, the joint strategy of the competitors changes as the number of competitors increases (see also the dependence on n in Theorem 1), resulting in a reduced joint profit of the competitors. Hence, the competitors have an incentive to collude. Such collusion could take on two forms. First, the n competitors can agree to each trade $1/n^{\text{th}}$ of the strategy of a single competitor. This way they optimize their proceeds as a group. However, each competitor has an incentive to deviate from the agreed strategy. Since all competitors need to be controlled at all points in time in order to avoid deviating, such an intensive collusion might be impossible. A second form of collusion requiring less control is an agreement about the asset position at the end of the first stage. When setting this target asset position at the right level, the competitors can increase their proceeds as a group while only controlling each other's asset position at one point in time.

2λ. Therefore the optimal asset position $X_i(T_1)$ is unchanged by multiplying both γ and λ by the same constant, i.e., it does not depend on γ and λ independently, but only on their ratio γ/λ. A similar reasoning shows that $X_i(T_1)$ is unchanged when T_1, T_2 and λ are multiplied by the same constant. This establishes the parameter dependence expressed by the function F in Equation 11. The exact functional form of the function F is very complicated and is therefore deferred to Appendix B.

The economic meaning of the ratio $\frac{\gamma}{\lambda}T_1$ respectively $\frac{\gamma}{\lambda}$ becomes most apparent by considering two polar market extremes:

- In **elastic markets** the major part of the initial total market impact vanishes after the execution of a market order (i.e., temporary impact $\lambda >>$ permanent impact γ). The market price in such markets behaves similar to an elastic rubber band: trading pressure can stretch it, but after the trading pressure reduces, the price recovers.

- In **plastic markets** the price impact of market orders is predominantly permanent (i.e., permanent impact $\gamma >>$ temporary impact λ). In such markets, the trading pressure exerts a "plastic deformation" on the market price.

Empirical studies report that the fraction $\frac{\gamma}{\lambda}$ takes a wide range of values and thus that markets are indeed sometimes plastic and sometimes elastic. Holthausen, Leftwich, and Mayers (1987) find that for their data sample, 75% of the total price impact of large transactions was temporary, while the follow-up study (Holthausen, Leftwich, and Mayers 1990) finds that for a different sample 85% of the total price impact was permanent. Coval and Stafford (2007) show that in markets where investors withdraw their money from open-ended mutual funds, the total price impact of transactions is predominantly temporary, while in other markets the price impact is predominantly permanent. The anecdotal evidence presented in the introduction indicates that the market for derivatives traded by LTCM was plastic, whereas the energy market was elastic during the Amaranth crisis. We devote the major part of this paper to the influence of the factor $\frac{\gamma}{\lambda}$ (respectively $\frac{\gamma}{\lambda}T_1$) on optimal strategies and liquidation proceeds.

By close inspection of Equation 12, we observe that in the case of a single competitor the sign of $X_1(T_1)$ is positive for small $\frac{\gamma}{\lambda}T_1$ (liquidity provision in elastic markets) and negative for large $\frac{\gamma}{\lambda}T_1$ (predation in plastic markets). This is a first indication of the richer dynamics in the two stage model compared to the one stage model. In the next section, we look into these relations in more detail.

C General properties

In this subsection we show that sunshine and stealth trading as well as predatory trading and liquidity provision can be rational, depending on the market parameters. In the subsequent Section IV, we discuss numerical examples that provide intuition for our results and illustrate the rich set of phenomena occurring in the two stage model.

C.1 Competitor behavior: Predatory trading versus liquidity provision

There are two facets of competitors' behavior: their trading during the first stage as a whole, which can be described by their asset position $X_i(T_1)$ at the end of the first stage, and their trading strategy within the first stage, which is given by $\dot{X}_i(t)$ for $0 \leq t \leq T_1$. We analyze these two facets one after the other, first for general n and thereafter for the special case of competitive markets ($n \to \infty$).

Theorem 3. *For all n, the asset position of the strategic players at the end of stage 1 is decreasing in $\gamma T_1/\lambda$:*

$$\frac{\partial}{\partial(\gamma T_1/\lambda)} X_i(T_1) = \frac{\partial}{\partial(\gamma T_1/\lambda)} F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) X_0 < 0. \tag{13}$$

In our two stage model, inter-stage cooperation is gradual. Theorem 3 shows that it is higher in elastic than in plastic markets: prices are more strongly dislocated by the seller's trades in elastic markets and therefore offer an attractive opportunity for liquidity provision. On the other hand, moving prices to the seller's disadvantage is cheaper in plastic markets, providing an incentive for predatory trading in such markets. In the repeated game model of Carlin, Lobo, and Viswanathan (2007), cooperation is not gradual, but binary: either competitors prey, or they refrain from preying. Carlin et al. obtain a result opposite to ours: In their model, cooperation is more likely in plastic markets than in elastic markets[17], since the benefits of cooperation grow with the permanent impact γ. Unfortunately, we are not aware of any empirical study of the drivers of cooperation and predation in real-world markets and therefore can only refer to the anecdotal examples given in the introduction.

The parameters T_2/T_1 and n have a complex effect on the optimal asset position of the predators. As we will see later in this section, monotonic relationships exist for competitive markets, i.e., as long as the number of competitors n is large. They break down for smaller numbers of n, as we will demonstrate in Section IV.

The monotonicity established in Theorem 3 raises the question of the minimum and maximum possible optimal asset position $X_i(T_1)$.

Proposition 4. *For all n, we have*

$$\lim_{\gamma T_1/\lambda \to 0} X_i(T_1) = \frac{1}{n+1}\frac{T_2 - T_1}{T_2} X_0 \tag{14}$$

$$\lim_{\gamma T_1/\lambda \to \infty} X_i(T_1) = -\frac{2}{n^3 + 4n^2 + n - 2} X_0 \tag{15}$$

The proposition establishes that in elastic markets, competitors provide the liquidity needed by the seller and later on unload their inventory to the general market. In plastic markets, competitors engage in predatory trading. In Section IV, we will provide explicit numerical examples in which liquidity provision and predatory trading occurs. Note that by Equation 15 the potential for predatory trading quickly decreases as the number of predators n increases.

Theorem 3 and Proposition 4 describe the asset position of the competitors at the end of the first stage. Although at least for small $\gamma T_1/\lambda$ we observe inter-stage cooperation, this does not necessarily imply that the seller is benefiting from the competitors' trading. Within the first stage, the trading of the competitors can significantly reduce her trading proceeds. In spite of *inter-stage cooperation*, there is still room for both *intra-stage predation* and *intra-stage cooperation*.

Proposition 5. *For all n, the trading rate of the competitors is increasing within the first stage:*

$$\dot{X}_i(t_2) > \dot{X}_i(t_1) \text{ for all } 0 \le t_1 < t_2 < T_1. \tag{16}$$

The initial trading rate is positive in purely elastic markets:

$$\dot{X}_i(0) = \frac{T_2 - T_1}{(n+1)T_1 T_2} X_0 > 0 \text{ for } \gamma T_1/\lambda = 0 \tag{17}$$

and decreasing in $\gamma T_1/\lambda$:

$$\frac{\partial}{\partial(\gamma T_1/\lambda)} \dot{X}_i(0) < 0. \tag{18}$$

By the previous proposition, we see that in elastic markets, competitors provide liquidity from the beginning of trading on already, i.e., they are *pure liquidity providers*. In moderate markets, competitors initially sell in parallel (*intra-stage predatory trading*), but subsequently buy back more than they initially sold short, resulting in a net liquidity provision by the competitors in the first stage. In plastic markets however the short-selling dominates and the competitors are overall predators during the first stage.

[17] See Result 4 and the remark thereafter in Carlin, Lobo, and Viswanathan (2007).

Corollary 6. *For all n and T_2/T_1, there are two threshold values $L \leq P \in]0, \infty]$ such that for $0 \leq \gamma T_1/\lambda \leq L$, competitors are pure liquidity providers, i.e.,*

$$\dot{X}_i(t) \geq 0 \text{ for all } 0 \leq t \leq T_1, \tag{19}$$

for $L < \gamma T_1/\lambda \leq P$, competitors engage in intra-stage predatory trading while being overall liquidity providers in the first stage, i.e.,

$$\dot{X}_i(0) \leq 0 \text{ and } X_i(T_1) \geq 0,$$

and for $P < \gamma T_1/\lambda$, competitors are overall predators in the first stage, i.e.,

$$X_i(T_1) < 0.$$

In the following we investigate the case of competitive markets ($n \to \infty$), for which we are able to derive simple closed form solutions as well as additional monotonicity relationships.

Proposition 7. *As the number of competitors n tends to infinity, the combined asset position of all competitors at the end of stage 1 converges to*

$$\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) = \lim_{n\to\infty} nX_1(T_1) = \frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}} - 1}{e^{\frac{\gamma(T_2)}{\lambda}} - 1} X_0. \tag{20}$$

In economic terms, this implies that for large n, some *inter-stage cooperation* between the seller and the competitors occurs regardless of the market parameters: since

$$\frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}} - 1}{e^{\frac{\gamma(T_2)}{\lambda}} - 1} > 0,$$

the competitors buy a portion of the seller's asset position in stage 1 and sell this portion in stage 2. The market thus functions efficiently: the competitors provide the liquidity required by the seller. It is important to stress that this is only the case when n is large. For small n, each competitor can influence market prices more easily since she is facing a smaller number of informed players that bring the market back into balance. "Selfish" strategies such as prolonged predatory trading thus remain attractive, as we have already remarked at the end of Subsection B.

We can draw an intuitive consequence of Proposition 7 for elastic markets: If the number of competitors is high, then the net sale of seller and competitors in each stage is proportional to the time available for selling. The following corollary expresses this in mathematical terms when sending λ to ∞.

Corollary 8. *As the number of competitors n and the temporary price impact coefficient λ tend to infinity, the combined asset position of all competitors at time T_1 converges:*

$$\lim_{\lambda\to\infty} \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) = \frac{T_2 - T_1}{T_2} X_0 \tag{21}$$

For general n, the competitors combined position $\sum_{i=1}^{n} X_i(T_1)$ at the end of the first stage is decreasing in $\gamma T_1/\lambda$, but not necessarily monotonous in the other two parameters T_2/T_1 and n. We will see examples of these relations in Section IV. The following proposition shows that these relations simplify in competitive markets ($n \to \infty$).

Corollary 9. *The proportion of the seller's asset position bought by the strategic players in stage 1 is given by*

$$\frac{\sum_{i=1}^{n} X_i(T_1)}{X_0} = nF\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right). \tag{22}$$

In the limit $n \to \infty$, the amount of liquidity provision depends on the drivers $\gamma T_1/\lambda$, T_2/T_1, and n in the following way:

$$\lim_{n\to\infty} \frac{\partial}{\partial(\gamma T_1/\lambda)} nF < 0 \qquad \lim_{n\to\infty} \frac{\partial}{\partial(T_2/T_1)} nF > 0 \qquad \lim_{n\to\infty} \frac{\partial}{\partial n} nF > 0. \tag{23}$$

In economic terms, the previous corollary states that for a large enough number n of competitors, the liquidity provision by strategic players in stage 1 is

- decreasing in $\gamma T_1/\lambda$,
- increasing in T_2/T_1, and
- increasing in n.

The first driver highlights the importance of the market environment; inter-stage cooperation is reduced in plastic markets. The second driver relates to the influence of risk management. If the competitors have enough capital, they will be willing to hold inventory for a long period of time, i.e., $T_2 >> T_1$. On the other hand, if they are in a financially weak condition, risk management is likely to limit the maximum holding period T_2 in order to reduce the associated risk. The third driver reflects the effect of limited competition among strategic players. By a combination of the latter two drivers, liquidity can disappear in a self-exciting vicious circle: Financial distress of some market participants can result in a general tightening of risk management practices and a smaller number of players engaging in strategic trading, leading to increased predatory trading and more distressed players.

Proposition 10. *In competitive markets ($n \to \infty$), the combined trading rate of the competitors at time 0 is*

$$\lim_{n\to\infty} \sum_{i=1}^{n} \dot{X}_i(0) = \frac{\gamma}{\lambda} \frac{2e^{\frac{\gamma}{\lambda}T_2} - e^{\frac{\gamma}{\lambda}(T_1+T_2)} - 1}{\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)\left(e^{\frac{\gamma}{\lambda}T_2} - 1\right)} X_0. \tag{24}$$

The competitors are pure liquidity providers, i.e.,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \dot{X}_i(t) > 0 \text{ for all } 0 \le t \le T_1 \tag{25}$$

if and only if

$$\frac{T_2}{T_1} > -\frac{\ln\left(2 - e^{\frac{\gamma}{\lambda}T_1}\right)}{\frac{\gamma}{\lambda}T_1}. \tag{26}$$

Otherwise, they engage in intra-stage predatory trading, i.e.,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \dot{X}_i(0) < 0.$$

Note that for large n, overall predation (the third option in Corollary 6) is not possible since competitors are always overall liquidity providers in the first stage (see Proposition 7). Figure 5 illustrates the regions of intra-stage cooperation and predation in competitive markets. Pure intra-stage cooperation is possible if the market is sufficiently elastic ($\frac{\gamma}{\lambda}T_1$ is small) and the competitors' time horizon T_2 is sufficiently larger than the seller's time horizon T_1. If the market is plastic, then the competitors will always initially sell in parallel to the seller, since the profit opportunity due to the expected price decline caused by the permanent impact is too attractive. In particular, if $\frac{\gamma}{\lambda}T_1 > \ln(2) \approx 0.69$, then intra-stage

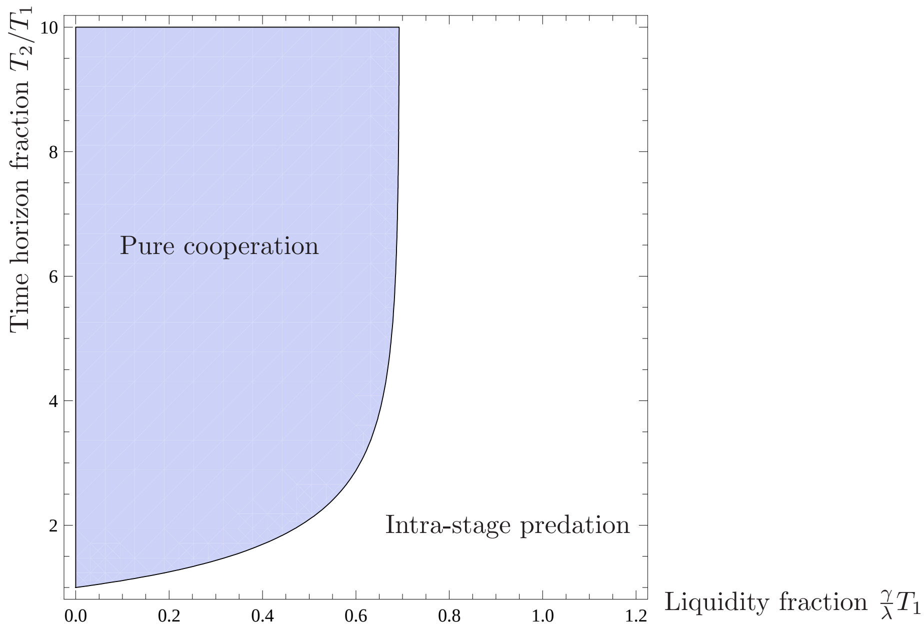

Figure 5: Cooperation versus intra-stage predation in competitive markets ($n \to \infty$) as a function of the fractions $\frac{\gamma}{\lambda}T_1$ and T_2/T_1. The shaded area corresponds to pure cooperation, the white area to intra-stage predation.

predation occurs irrespective of the competitors' time horizon. On the other hand, in the limit of a completely elastic market, we obtain

$$\lim_{\frac{\gamma}{\lambda}T_1 \to 0} -\frac{\ln\left(2 - e^{\frac{\gamma}{\lambda}T_1}\right)}{\frac{\gamma}{\lambda}T_1} = 1. \tag{27}$$

Equation 26 is thus fulfilled for all $T_2 > T_1$, and we obtain pure liquidity provision by the competitors irrespective of their time horizon T_2. For intermediate markets, the competitors' time horizon T_2 influences the sign of the initial trading speed of the competitors. A shorter time horizon T_2 gives the competitors less time to unwind any long positions they obtained during the first stage. For small T_2/T_1, the model is thus similar to the one-stage model for which we expect to see intra-stage predation.

C.2 Seller behavior: Stealth versus sunshine trading

We now turn to the return that the seller can expect to receive in a market with a certain number n of strategic competitors.

Theorem 11. *By selling an asset position X_0 in stage 1, the seller receives an average total cash position of*

$$R_0 = X_0\left(P_0 - \gamma X_0 G\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right)\right). \tag{28}$$

The function G is given in closed form in the proof in Appendix B. For large n, the seller's return is

- *increasing in $\gamma T_1/\lambda$,*
- *increasing in T_2/T_1, and*

- *increasing in n.*

It converges to:

$$\lim_{n\to\infty} R_0 = X_0\left(P_0 - \gamma X_0 \frac{1}{1-e^{-\frac{\gamma}{\lambda}T_2}}\right) \tag{29}$$

Given the result above, the benefits of sunshine trading can easily be quantified. If the seller's intentions remain secret, she can expect a return of

$$X_0\left(P_0 - \gamma X_0/2 - \lambda X_0/T_1\right). \tag{30}$$

Alternatively, she can pre-announce her intentions, attract a large number of competitors and thus expect a return of

$$X_0\left(P_0 - \gamma X_0 \frac{1}{1-e^{-\frac{\gamma}{\lambda}T_2}}\right). \tag{31}$$

Proposition 12. *Assuming that pre-announcement of sales attracts a large number of competitors ($n \to \infty$), sunshine trading is superior to stealth trading if*

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1-e^{-\frac{\gamma}{\lambda}T_2}}. \tag{32}$$

If the competitors do not face any material time constraint ($T_2 \to \infty$), sunshine trading is beneficial if

$$\frac{\gamma}{\lambda}T_1 < 2. \tag{33}$$

For the case of an infinite number of potential competitors who face no material time constraint ($n \to \infty$ and $T_2 \to \infty$), we can differentiate three cases by Propositions 10 and 12:

- For $0 \le \frac{\gamma}{\lambda}T_1 \le \ln 2$, informed competitors provide liquidity during the entire first stage. Sunshine trading is thus obviously the optimal execution strategy for the seller.

- For $\ln 2 < \frac{\gamma}{\lambda}T_1 < 2$, competitors first sell in parallel to the seller, however quickly buy back and provide liquidity, resulting in an overall increase in expected liquidation proceeds for the seller and thus an incentive to sunshine trade.

- For $\frac{\gamma}{\lambda}T_1 > 2$, the negative effects of intra-stage predation outweigh the positive effects of inter-stage cooperation, and the seller is better off by stealth trading.

The same differentiation into three cases also holds in competitive markets when the competitors face a material time constraint ($n \to \infty$, but $T_2 << \infty$). However the algebraic expressions for the boundaries for $\frac{\gamma}{\lambda}T_1$ are not as simple and depend on the time constraint T_2. For the case of a finite number of competitors $n << \infty$, the above characterization becomes significantly more difficult. The choice of sunshine and stealth execution then depends not only on the fraction $\frac{\gamma}{\lambda}T_1$, but also on the number n of competitors that can be attracted by preannouncement. Due to the complexity of the resulting formulas, we limit our analysis of the case $n << \infty$ to numerical examples in Section IV.

In our model, the market liquidity parameters γ and λ and the length of the two stages T_1 and $T_2 - T_1$ determine whether sunshine trading is beneficial. These drivers are not relevant in existing models. Most notably, sunshine trading is always beneficial in the model used by Admati and Pfleiderer (1991), while it is never beneficial in equilibrium in the model of Brunnermeier and Pedersen (2005).

In the previous discussion, we assumed that pre-announcing a trade does not change market-wide liquidity. In case sunshine traders are structurally special, this assumption can be weakened by changing λ and γ for sunshine trades. For example, Admati and Pfleiderer (1991) assume that sunshine traders are uninformed; their trades should therefore result in a smaller (or possibly even no) permanent price change. This can be incorporated by assuming a smaller value of γ for sunshine trades.

C.3 Price evolution

We now analyze the market prices resulting from the combined trading activities of the seller and the competitors in more detail. When trading commences in $t = 0$, the expected price jumps downward from its level $\bar{P}(0-) = P_0$ to $\bar{P}(0) = P_0 + \lambda \sum_{i=0}^{n} \dot{X}_i(0)$ due to the temporary impact of the selling. After the initial price jump, the expected price $\bar{P}(t)$ is exhibiting a downward trend in equilibrium (see Section IV). This indicates that our model market does not fulfill the strong form of the efficient markets hypothesis as introduced by Fama (1970): if relevant information is shared by only a *small* number of market participants, then this information is only slowly reflected in market prices. On the other hand, empirical evidence suggests that capital markets are efficient in the semi-strong sense. We would therefore expect that if the seller's intentions are known by a *sufficiently large* number of market participants, this information is instantaneously fully reflected in market prices. Public information can thus not be used to predict price changes. The following proposition states that this is indeed the case in our market model.

Proposition 13. *The absolute value of the drift $|\dot{\bar{P}}(t)|$ is a decreasing function of the number of competitors n. In the limit, the expected market price instantaneously jumps to*

$$P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma(T_2)}{\lambda}}} X_0 \tag{34}$$

and is constant from thereon throughout stage 1 and stage 2 until the end of stage 2.

The preceding proposition yields an interpretation for the asymptotic expected return of the seller derived in Theorem 11: the asymptotic expected return in Equation 29 is equal to the initial asset position X_0 times the asymptotic expected price in Equation 34.

In plastic markets, the initial price jump $|\bar{P}(0) - P_0|$ is an increasing function of the number n of competitors, while it is a decreasing function of n in elastic markets. It is interesting to note that the new equilibrium price $P_0 - \frac{\gamma}{1-e^{-\frac{\gamma(T_2)}{\lambda}}} X_0$ does not depend on whether the seller can trade in stage 2 (see Proposition A.1). When discussing numerical examples in Section IV, we also give an intuitive explanation of the forces that ensure the semi-strong efficiency in our model.

To formally discuss price overshooting, we include the time after T_2 in our analysis, i.e., the time after the seller and the competitors have stopped trading. The temporary impact of the trades during $[0, T_2]$ vanishes immediately at T_2; therefore, only the permanent impact remains. The seller sold X_0 while the competitors did not change their asset positions. Therefore we obtain an expected market price of $\bar{P}(T_2+) = P_0 - \gamma X_0$ for the time after T_2. If during the trading phase $[0, T_2]$ the price drops below $\bar{P}(T_2+)$, i.e.,

$$\min_{t \in [0,T_2]} \bar{P}(t) - \bar{P}(T_2+) < 0 \tag{35}$$

we say that the price *overshoots*. We can now describe the relationship between price overshooting and predatory activity.

Proposition 14. *The price $\bar{P}(t)$ attains its minimum in the interval $[0, T_2]$ at the end of the first stage:*

$$\min_{t \in [0,T_2]} \bar{P}(t) = \bar{P}(T_1-) \tag{36}$$

Price overshooting occurs irrespective of the presence of competitors:

$$\bar{P}(T_1-) < \bar{P}(T_2+). \tag{37}$$

The level of price overshooting $\bar{P}(T_2+) - \bar{P}(T_1-)$ is increased by competitors only in very plastic markets, i.e., only if the permanent impact is much larger than the temporary impact. In all other circumstances, price overshooting is reduced by competitors. If competitors are already active in the market ($n \geq 1$), then additional competitors reduce price overshooting irrespective of the market character.

Parameter	Elastic market	Plastic market	Intermediate market
Asset position X_0		1	
Initial price P_0		10	
Duration T_1 of stage 1		1	
Duration $T_2 - T_1$ of stage 2		1	
Permanent impact sensitivity γ	1	3	1.8
Temporary impact sensitivity λ	3	1	1

Table II: Parameter values used for numerical computation in Section IV.

It is interesting to compare our results to the models introduced by Brunnermeier and Pedersen (2005) and by Carlin, Lobo, and Viswanathan (2007). Preying introduces price overshooting in the first framework, but it reduces price overshooting in the latter (see Proposition A.2); in our model, the effect of preying on price overshooting depends on the market. In all three models, price overshooting is reduced by additional competitors (assuming that at least one competitor is active).

IV Example markets

In this section, we numerically analyze the optimal strategies for the seller and the competitors and their impact on the proceeds of the seller and on the market price. We focus on the qualitative influence of the ratio $\frac{\gamma T_1}{\lambda}$ and of the number of competitors n. For notational simplicity, we will assume that $T_1 = 1$, $T_2 = 2$ and thus restrict our discussion to the impact of the parameters γ, λ and n.

We first investigate the two polar market extremes of elastic and plastic markets (see Section III.B for a definition). In many practical cases, the market will fall into neither of these two categories. Instead temporary and permanent impact will be balanced. We therefore conclude our case analysis by reviewing an intermediate market, that is, a market where temporary and permanent impact are balanced: $\lambda \approx \gamma$. For the numerical computations, we used the parameter values given in Table II.

A Example market 1: Elastic market

To begin with, let us assume that no competitors are active in the market. In such a situation, it is optimal for the seller to sell her asset position linearly (Figure 6). We therefore expect that the market price in stage 1 drops dramatically (Figure 7), since in order to satisfy the seller's trading needs, liquidity is required fast—which is expensive in an elastic market. In stage 2, the seller no longer generates temporary impact, and the price bounces back. Furthermore, since the permanent impact is comparatively small, the price recovers almost completely.

A competitor knowing of the seller's intentions would expect this price pattern. Her natural reaction would therefore be to buy some of the seller's shares in stage 1 at a low price and to sell them in stage 2 at a higher price. Figure 8 shows that this is indeed what happens when the seller and the competitors follow their optimal strategies. In our example, the competitors are pure liquidity providers: they are buying throughout the entire first stage, in line with Proposition 10.

As can be seen in these figures, the relationship established for large n in Corollary 5 holds for all n in our example: The total asset position $\sum_{i=1}^{n} X_i(T_1)$ acquired by the competitors at the end of stage 1 increases as the number of competitors increases (see also Figure 9). To gain some intuition for this phenomenon, let us assume that n_1 competitors optimally acquire a joint asset position of $n_1 Y_1$ shares. Imagine one of the competitors increases her target asset position by 1. This will decrease the profit per share that she makes, but adds another share to her profitable portfolio. If the original target position

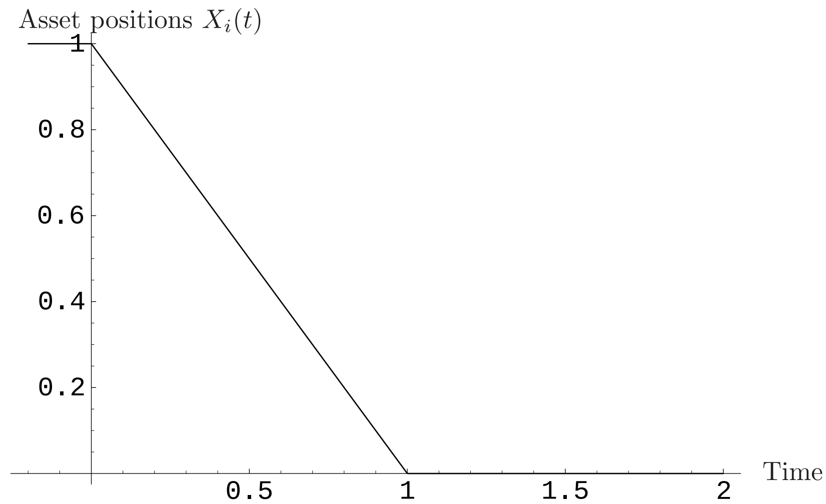

Figure 6: Asset position $X_0(t)$ of the seller when no competitors are active.

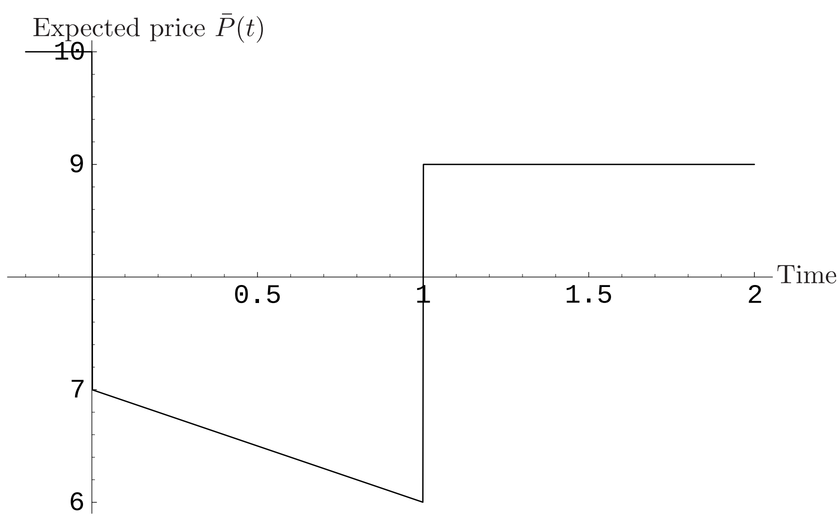

Figure 7: Expected price $\bar{P}(t)$ in an elastic market over time when no competitors are active; at time $t = 1$, stage 1 ends and stage 2 begins.

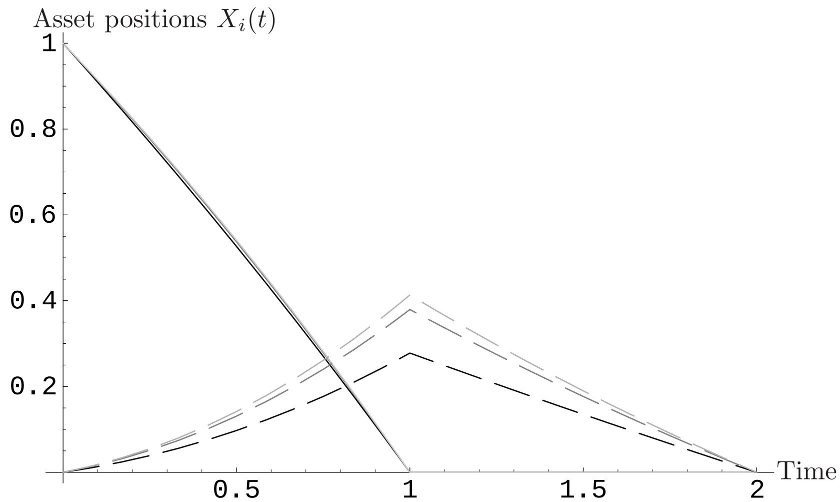

Figure 8: Asset positions $X_i(t)$ over time in an elastic market; at time $t = 1$, stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all n competitors. The black lines correspond to $n = 2$, the dark grey lines to $n = 10$ and the light grey lines to $n = 100$.

Y_1 is optimal, then this increase will leave her total profit roughly unchanged:

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times Y_1 \approx 0. \tag{38}$$

Let us now assume that $n_2 > n_1$ competitors are active and that they jointly acquire $n_1 Y_1$ shares. Now, increasing the target position $\frac{n_1 Y_1}{n_2}$ of an individual competitor by one share changes the competitor's total profit by

$$\text{Profit per share} \times 1 - \text{Decrease in profit per share} \times \frac{n_1 Y_1}{n_2} > 0. \tag{39}$$

Therefore each competitor has an incentive to increase the trading target for the end of stage 1, resulting in an increased joint trading target.

The effect of the competitors' trading (buying in stage 1, selling in stage 2) is that prices between stage 1 and stage 2 will even out, as predicted by Propositions 13 and 14: The large price jumps expected in the absence of competitors will disappear if the number of competitors is large enough (see Figure 10). The price overshooting created by the selling pressure of the seller is therefore reduced by the competitors.

From the seller's perspective, the competitors' trading is beneficial; by buying some of her shares, the competitors reduce the seller's market impact and thus increase her return. As we have just discussed, a larger number of competitors implies a larger combined purchase by the competitors. Hence, the seller can expect to profit from each additional competitor, i.e., the larger the number of competitors, the larger her profit. This is illustrated by Figure 11; the seller's return is higher when competitors are active than it is when there are no competitors. The monotonicity with respect to n established in Theorem 11 therefore holds for all n in this example.

The practical implications are evident: in an elastic market, it is sensible to announce any large, time-constrained asset transaction directly at the beginning of trading in order to attract liquidity.

B Example market 2: Plastic market

We will now turn to plastic markets, i.e., markets with a permanent impact that considerably exceeds the temporary impact. In such a setting, we expect the price dynamics to be very different from the

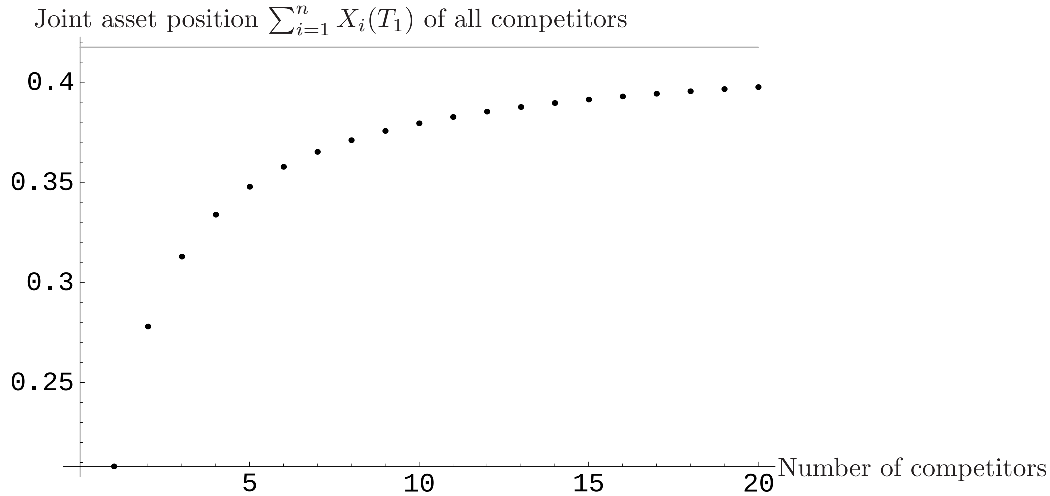

Figure 9: Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors in an elastic market at time T_1 depending on the total number n of all competitors. The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$.

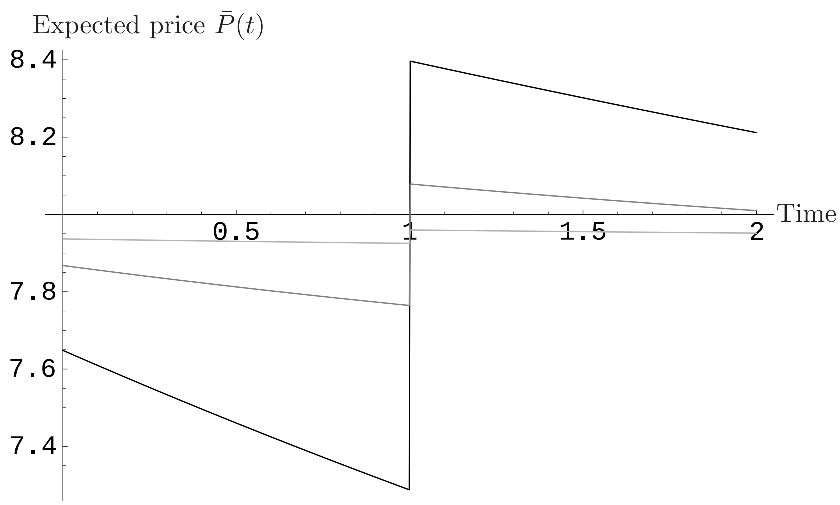

Figure 10: Expected price $\bar{P}(t)$ in an elastic market over time depending on the number of competitors n; at time $t = 1$, stage 1 ends and stage 2 begins. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. A significant reduction in price drift can be observed; furthermore, $\bar{P}(0)$ is smaller than $P_0 = 10$.

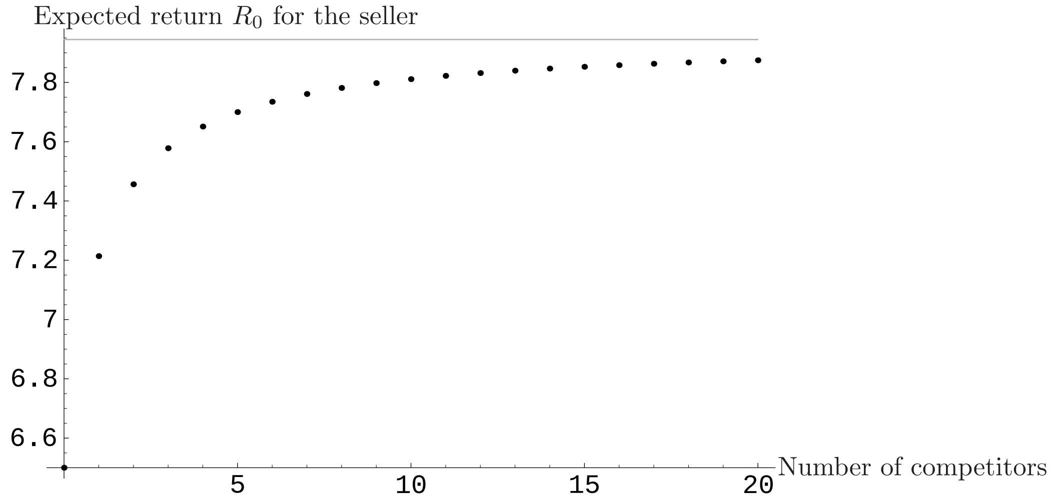

Figure 11: Expected return R_0 for the seller in an elastic market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

dynamics described for elastic markets in the previous subsection.

Let us again assume that no competitors are active. Then the optimal trading strategy for the seller is again a linear decrease of the asset position (see Figure 6). In stage 1, the seller is constantly pushing the market price further and further down; we therefore expect the price to be high at the beginning of stage 1 and low at the end of stage 1 (see Figure 12). In stage 2, the price will bounce back, since the temporary impact of the seller's trading has vanishes. However, this jump will be comparatively small because the temporary price impact is small.

For a competitor, this implies that buying some of the seller's shares in stage 1 does not promise any large profit; the price reversion in stage 2 is too small. Instead, it appears more profitable to exploit the price change *within* stage 1 rather than *between* stage 1 and stage 2. By selling short the asset at the beginning of stage 1 and buying it back at the end of stage 1, she can likely make a large profit. Thus, we expect to see preying behavior similar to the behavior in the one stage framework discussed in Section II. Our hypothesis is verified by the numerical results shown in Figure 13.

We observe that for all n there is intra-stage predation. The asset position $X_i(T_1)$ of the competitors at the end of the first stage however changes from a short position to a long position as the number of competitors increases. This change from inter-stage predation to inter-stage cooperation is required by Corollary 9 and can be explained intuitively in the following way. For a small number of competitors the price evolution will be sufficiently close to the one shown in Figure 12, therefore preying is attractive and the competitors will enter stage 2 with a short position. As the number of competitors increases, the price curve flattens within the first stage due to the increased competition for profit from predatory trading (Figure 14; see also Proposition A.1 in the appendix). Therefore the recovery of prices between stage 1 and stage 2 now becomes attractive, even though it is relatively small. Similar to the line of argument in elastic markets, it now pays off for the competitors to acquire a small asset position toward the end of stage 1 in order to sell it during stage 2. This is illustrated in Figure 15. If the number of competitors is small, it is beneficial to enter stage 2 with a short position; if the number of competitors is large, it is more attractive to enter stage 2 with a long position.

Based on this line of argument, we expect the price overshooting to disappear if the number of competitors is large. A single competitor however can decrease or increase price overshooting, depending

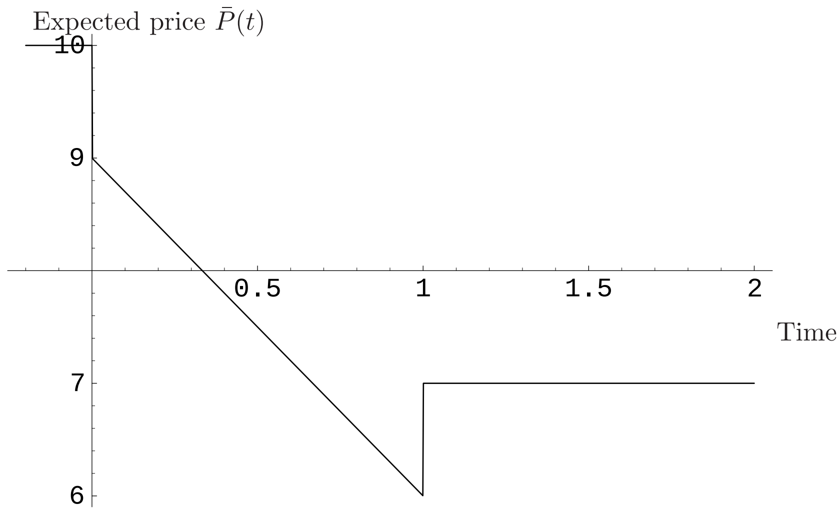

Figure 12: Expected price $\bar{P}(t)$ in a plastic market over time when no competitors are active; at time $t = 1$, stage 1 ends and stage 2 begins.

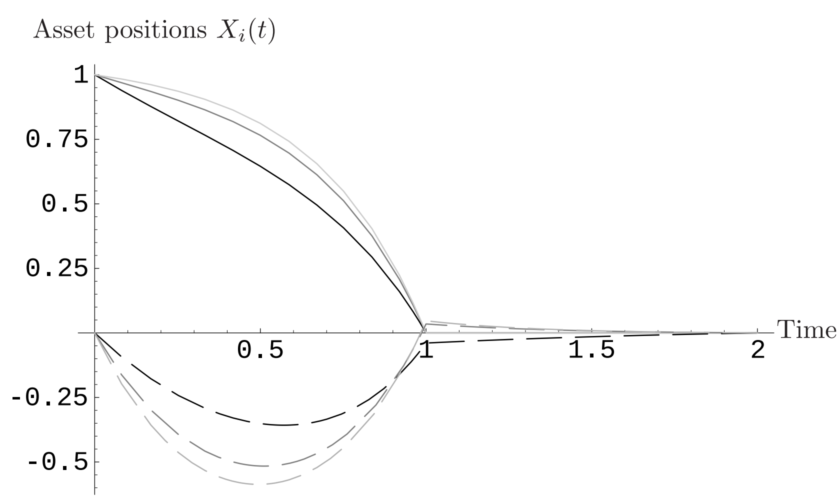

Figure 13: Asset positions $X_i(t)$ over time in a plastic market; at time $t = 1$, stage 1 ends and stage 2 begins. The solid lines represents the seller, the dashed lines the combined asset position of all n competitors. The black lines correspond to $n = 2$, the dark grey lines to $n = 10$ and the light grey lines to $n = 100$.

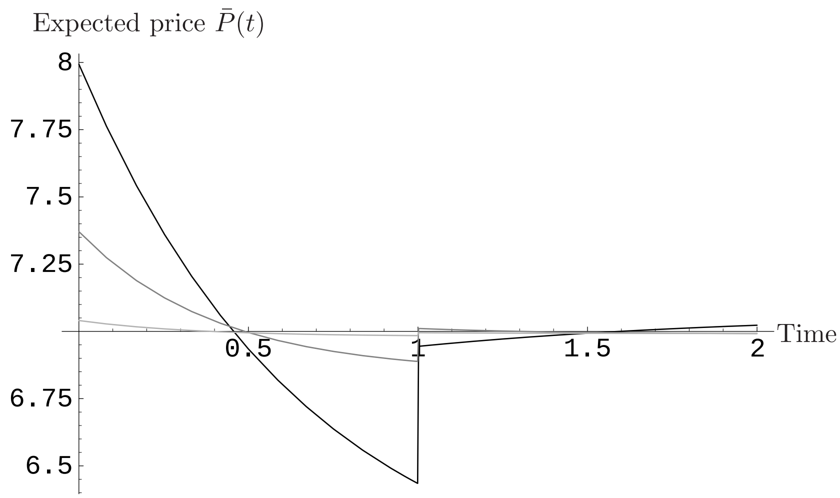

Figure 14: Expected price $\bar{P}(t)$ in a plastic market over time depending on the number of competitors n; at time $t = 1$, stage 1 ends and stage 2 begins. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. A significant reduction in price drift can be observed.

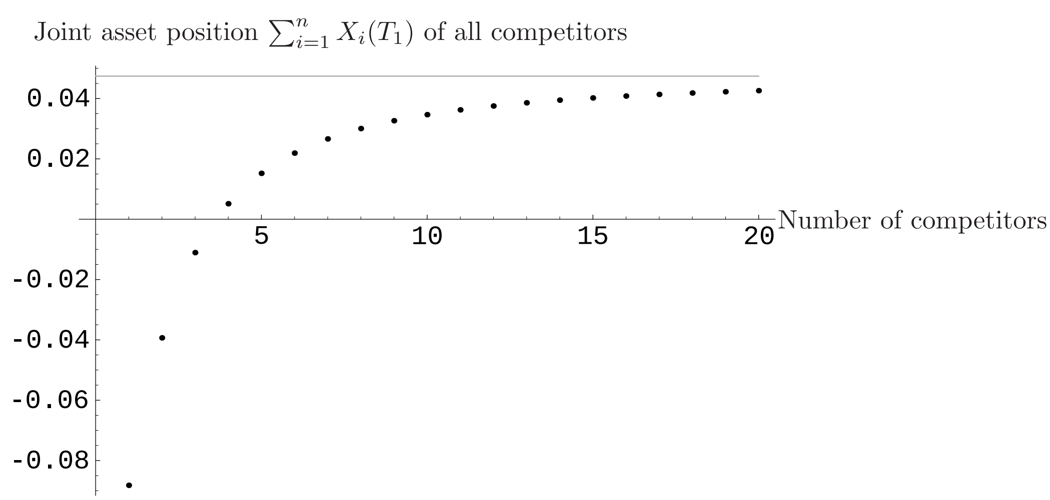

Figure 15: Joint asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors in a plastic market at time T_1 depending on the total number n of all competitors. The grey line represents the limit $\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1)$.

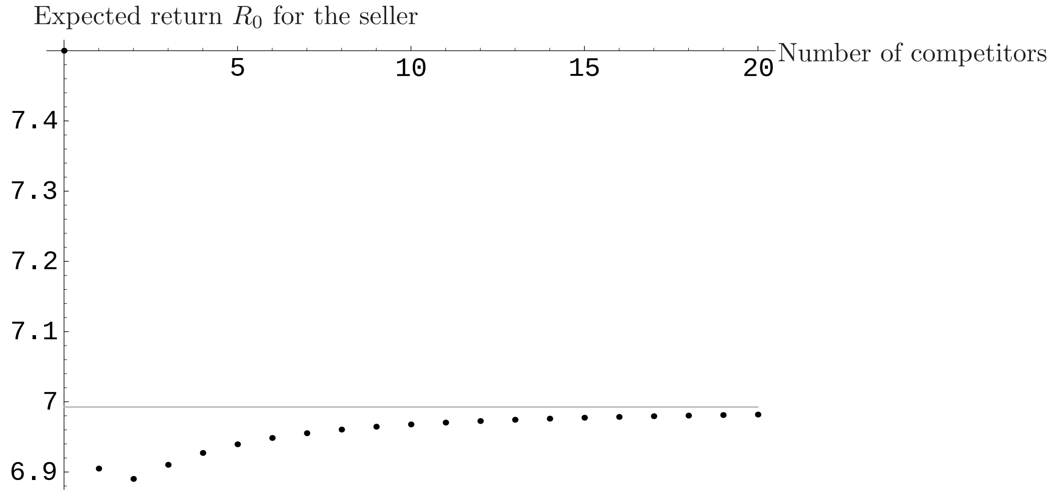

Figure 16: Expected return R_0 for the seller in a plastic market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

on how plastic the market is. In the plastic market considered in this section, even a single competitor reduces price overshooting. But if the permanent impact is increased to 7.0 and all other parameters are unchanged, a single competitor increases price overshooting.

Similar to the results of Section II, we might be tempted to expect that the return for the seller decreases as the number of competitors increases and predation becomes more fierce. Figure 16 shows that this is not the case and instead confirms the validity of Theorem 11. The return for the seller is significantly decreased by competitors; furthermore, two competitors decrease it more than a single competitor. However, the return for the seller is higher when three competitors are active than when only two competitors are active; as soon as at least two competitors are active, each additional competitor is beneficial for the seller.

The connection between the return for the seller and the number of competitors is a combination of effects from the one stage model and the two stage model in an elastic market. The first effect (already observed in the one stage model) is that a larger number of competitors leads to more aggressive preying and hence to a reduced return for the seller. This effect is very strong for a small number of competitors, but not for a large number of competitors. The second effect is that a larger number of competitors also results in an increased total asset position $\sum_{i=1}^{n} X_i(T_1)$ of all competitors at the end of stage 1. This reduces the trading pressure in stage 1 and therefore increases the return for the seller. The latter effect dominates the first if the number of competitors is large. This illustrates that the monotonicity established in Theorem 11 for large n can break down for small n.

C Example market 3: Intermediate market

In most cases, the differences between the temporary and permanent impact factors γ and λ will not be as extreme as depicted in the two previous cases. If the two parameters are closer together, we can expect to observe characteristics of both elastic as well as plastic markets:

- At the beginning of the first stage, the competitors "race the seller to market", that is, they sell in parallel to her (intra-stage predation).

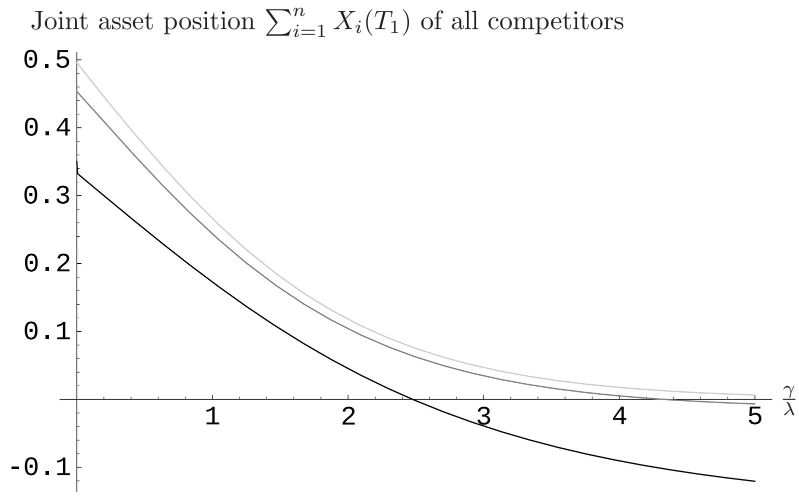

Figure 17: Asset position $X_1(T_1)$ of the competitors, depending on $\frac{\gamma}{\lambda}$. The black line corresponds to $n = 2$, the dark grey line to $n = 10$ and the light grey line to $n = 100$. The other parameters are chosen as in Table II.

- For a small number of competitors, the competitors end the first stage with either a long or a short position depending on whether the market is more elastic or more plastic (see Figure 17).
- For a large number of competitors, the competitors buy back more shares than they sold at the beginning of stage 1 (inter-stage cooperation).
- If the number of competitors is large, then price overshooting is reduced and market prices are almost flat and almost the same in stage 1 and stage 2.
- If a certain minimum number of competitors is active, then additional competitors increase the return for the seller since the increase in inter-stage cooperation outweighs the increase in intra-stage predation.

One interesting question remains open so far. We have already seen that in elastic markets the seller benefits from competitors, whereas in plastic markets the seller prefers to have no competitors at all. What is the situation in an intermediate market? Of course, both effects may apply depending on whether the market is more plastic or more elastic in nature. However, a new phenomenon can also arise: It might be the case that a small number of competitors is harmful to the seller's profits, but a large number increases the profits even beyond the case of no predation (see Figure 18 for an example).

The practical implications are evident: If there are already some informed traders or if the seller expects to be able to attract a sufficient number of competitors, announcing her trading intentions can be attractive; if there is only a limited number of potential competitors she is best advised to conceal her intentions.

V Summary and Conclusions

In a number of practical cases, investors need to liquidate large asset positions in a short time. In this paper, we describe optimal liquidation strategies in case other market participants are aware of

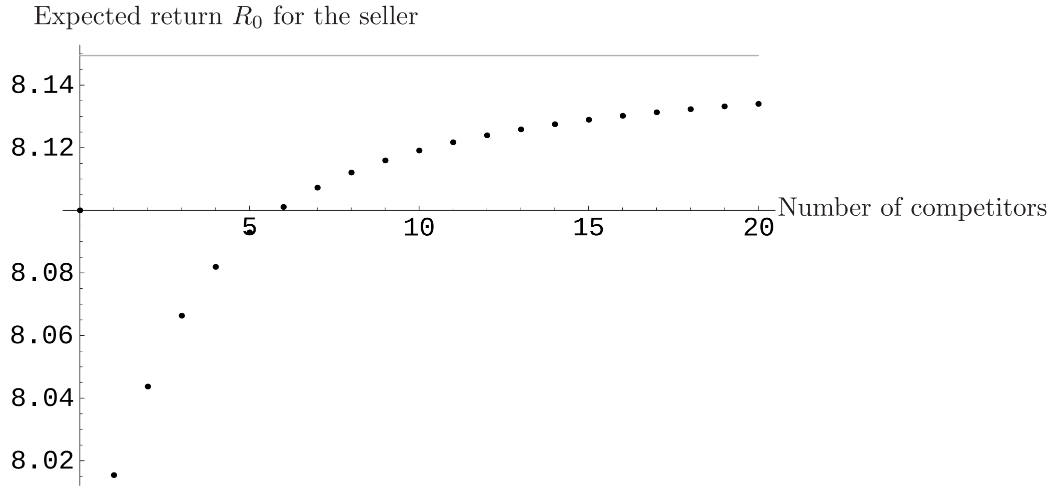

Figure 18: Expected return R_0 for the seller in an intermediate market, depending on the number of competitors. The grey line represents the limit $n \to \infty$. The return for the seller without competitors is at the intersection of x- and y-axis.

the investor's needs. A crucial assumption is that these competitors are not limited by the same time constraint the seller is facing.

We solve a competitive trading game in an illiquid market model incorporating a temporary and a permanent price impact. Each player faces a dynamic programming problem. According to our model, the optimal strategies for these competitors depend on the liquidity characteristics of the market. If the permanent impact affects market prices more heavily than the temporary impact, the competitors will "race" the seller to market, selling in parallel with her and buying back after the seller sold her asset position. If price impact is predominantly temporary, competitors provide liquidity to the seller by buying some of her shares and selling them after the seller has finished her sale. In the first case, the seller should conceal her trading intentions in order not to attract competitors, while in the latter case, pre-announcing a trade can attract liquidity suppliers and thus be beneficial.

As a special case, we investigate behavior in a market with a very large number of competitors. We find that in spite of illiquidity, such a market efficiently determines a new price. Information about the seller's intentions is immediately incorporated into the market price and does not affect it thereafter. The competitors might race the seller to market, but even in markets with high permanent impact, they quickly start buying back shares and sell these after the seller has finished her sale.

In conclusion, we believe that our analysis enhances the understanding of stealth and sunshine trading as well as liquidity provision and predation in the marketplace.

A Propositions on the one stage model

We first state two propositions concerning the one stage model introduced in Section II. These are used for comparison of the one stage model and the two stage model as well as in the proofs presented in Appendix B.

Proposition A.1. *In the one stage model, the absolute value of the drift $|\dot{P}(t)|$ is a decreasing function* *of n. In the limit case $n \to \infty$, the expected market price instantaneously jumps to*

$$P_0 - \frac{\gamma}{1 - e^{-\frac{\gamma T_1}{\lambda}}} X_0 \tag{40}$$

and is constant from thereon until the end of trading at time $t = T_1$.

Proof of Proposition A.1. Using the notation from Theorem 1, the combined trading speed of the seller and all competitors amounts to

$$\sum_{i=0}^{n} \dot{X}_i(t) = \sum_{i=0}^{n} (a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} + b_i e^{\frac{\gamma}{\lambda}t}) = (n+1) a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}. \tag{41}$$

The change in combined asset position at time t is therefore:

$$\sum_{i=0}^{n} (X_i(t) - X_i(0)) = \sum_{i=0}^{n} \int_0^t \dot{X}_i(s) ds = \int_0^t \sum_{i=0}^{n} \dot{X}_i(s) ds \tag{42}$$

$$= \int_0^t (n+1) a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}s} ds = (n+1) \frac{n+2}{n} \frac{\lambda}{\gamma} a \left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}\right) \tag{43}$$

Now, we can compute the expected market price:

$$\bar{P}(t) = P_0 + \gamma \sum_{i=0}^{n} (X_i(t) - X_i(0)) + \lambda \sum_{i=0}^{n} \dot{X}_i(t) \tag{44}$$

$$= P_0 + \gamma(n+1) \frac{n+2}{n} \frac{\lambda}{\gamma} a (1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}) + \lambda(n+1) a e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t} \tag{45}$$

$$= P_0 + \lambda \frac{n+1}{n} (n + 2 - 2e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}) a \tag{46}$$

$$= P_0 + \lambda \frac{n+1}{n} (n + 2 - 2e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}) \frac{n}{n+2} \frac{\gamma}{\lambda} \left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}\right)^{-1} \frac{-X_0}{n+1} \tag{47}$$

$$= P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} + \gamma X_0 \frac{2}{n+2} \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \tag{48}$$

Only the last term in the expression above is time dependent; its influence decreases with increasing n. In the limit, we obtain that the expected market price $\bar{P}(t)$ is constant:

$$\lim_{n\to\infty} \bar{P}(t) \equiv P_0 - \gamma X_0 \frac{1}{1 - e^{-\frac{\gamma}{\lambda}T_1}} \tag{49}$$

□

Proposition A.2. *Without any competitors (i.e., nobody is aware of the seller's intentions), the price overshoots by $\lambda X_0/T_1$. If competitors are present, the price overshooting is reduced to*

$$\frac{n}{n+2} \gamma X_0 \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}, \tag{50}$$

which is a decreasing function of the number n of competitors.

Proof of Proposition A.2. Without any competitors, the optimal strategy for the seller is to liquidate her asset position linearly: $X_0(t) = (T_1 - t) X_0/T_1$. The market price thus drops to

$$\bar{P}(T_1-) = P_0 - \gamma X_0 - \lambda X_0/T_1. \tag{51}$$

and price overshooting amounts to $\lambda X_0/T_1$.

From Equation 46, we know the structure of $\bar{P}(t)$ when competitors are present and deduce that the market price decreases to

$$\bar{P}(T_1) = P_0 - \gamma X_0 \frac{1}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} + \gamma X_0 \frac{2}{n+2}\frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}. \quad (52)$$

Thus, the price overshoots with magnitude

$$\bar{P}(T_1) - \bar{P}(T_1-) = \frac{n}{n+2}\gamma X_0 \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}{1-e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}. \quad (53)$$

The monotonicity follows directly. □

B Proofs for propositions on the two stage model

The proofs of the theorems, propositions and corollaries presented in this paper are given in order of appearance in the main body of text. In order to keep the proofs compact, they sometimes use results that are independently proven later in this appendix.

Proof of Theorem 2. The actual computations are lengthy; we will therefore only sketch the approach (more details are available from the authors on request).

Let us first discuss the case $n = 1$, i.e., the seller is facing only one competitor. By computations similar to the ones in Proposition A.1, we can express the expected market price $\bar{P}(t)$ as a *linear* function of the seller's asset position X_0 and the competitors asset position $X_1(T_1) = Z_1$ at the end of stage 1. Furthermore, by Theorem 1 the competitor's trading speed $\dot{X_1}(t)$ is linear in X_0 and Z_1. Therefore we can then calculate the return for the competitor in the two stages as *quadratic* functions of X_0 and Z_1:

$$Return_{Competitor} = Return_{Stage1}(X_0, Z_1) + Return_{Stage2}(X_0, Z_1) \quad (54)$$

Now, we can determine the optimal Z_1 by maximizing the quadratic function $Return_{Competitor}$, i.e., by determining the root of its derivative, which is a linear function in X_0. Thereby we obtain Equation 12.

Let us turn to the case $n \geq 2$, i.e., the seller is facing at least two competitors. We assume that $n-1$ competitors acquire optimal asset positions $X_i(T_1) = Y_i$ for $1 \leq i \leq n-1$ and solve for the optimal asset position $X_n(T_1) = Z_n$ for the last competitor. Similar to the case $n = 1$ discussed above, we can calculate the return for the last competitor as a *quadratic* function of $X_0 + \sum_{i=1}^{n-1} Y_i$ and Z_n:

$$Return_{Competitor_n} = Return_{Stage1}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n) + Return_{Stage2}(X_0 + \sum_{i=1}^{n-1} Y_i, Z_n) \quad (55)$$

We can again determine the optimal Z_n by maximizing $Return_{Competitor_n}$ and obtain a *linear* function of $X_0 + \sum_{i=1}^{n-1} Y_i$:

$$Z_n^{optimal} = f(X_0 + \sum_{i=1}^{n-1} Y_i) \quad (56)$$

Similarly we obtain the linear equations

$$Z_j^{optimal} = f(X_0 + \sum_{i=1, i\neq j}^{n} Y_i) \quad (57)$$

for all $1 \le j \le n$. Since we assumed that $(Y_1, \dots, Y_n)$ was optimal in the first place, we know that the optimal $Z_j^{optimal}$ has to be equal to Y_j; we therefore obtain

$$Y_j = f(X_0 + \sum_{i=1, i\neq j}^{n} Y_i) \tag{58}$$

for all $1 \le j \le n$. The set of linear equations (58) constitutes a symmetric, non-singular linear problem of n equations in n variables. Its unique solution therefore has to fulfill $Y_1 = \cdots = Y_n$ and these Y_i are a linear function of X_0. By computing this linear function precisely, we obtain the functional form

$$F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) = -\frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} \tag{59}$$

with parameters

$$A_0 = 2\Bigg(- e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} - e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}\Bigg)$$

$$A_1 = 3e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 3e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + 3e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - 2e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} - e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + 2e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}} + 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}} - 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}$$

$$A_2 = e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{\left(2+3n+n^2\right)\lambda}} - e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{\left(2+3n+n^2\right)\lambda}} + e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{\left(2+3n+n^2\right)\lambda}}$$

and

$$
\begin{aligned}
B_0 = -2\Bigg(& 2e^{\frac{(1+2n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - \\
& e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma(n(3+2n)T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - \\
& e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} - \\
& 2e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + 2e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
B_1 = {} & 2e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} - e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} + \\
& e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - \\
& 2e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - 2e^{\frac{\gamma\left(-(4+3n)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + \\
& 2e^{\frac{\gamma T_1+n\gamma T_2}{\lambda+n\lambda}}
\end{aligned}
$$

$$
\begin{aligned}
B_2 = 2\Bigg(& e^{\frac{(2+n)\gamma(-T_1+T_2)}{(1+n)\lambda}} - 2e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(2+2n+n^2\right)T_1+n(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + \\
& 2e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(-\left(2+4n+n^2\right)T_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{n\gamma T_1+\gamma T_2}{\lambda+n\lambda}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
B_3 = {} & -e^{\frac{\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{n\gamma(-T_1+T_2)}{\lambda+n\lambda}} + e^{\frac{\gamma(-T_1+(2+n)T_2)}{(1+n)\lambda}} - e^{\frac{n\gamma(-T_1+(2+n)T_2)}{(1+n)(2+n)\lambda}} + \\
& e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma\left(\left(-2+n^2\right)T_1+(2+n)^2T_2\right)}{(1+n)(2+n)\lambda}} - e^{\frac{\gamma(-nT_1+(1+2n)T_2)}{(1+n)\lambda}} + \\
& e^{\frac{\gamma\left(-nT_1+\left(2+5n+2n^2\right)T_2\right)}{(1+n)(2+n)\lambda}}
\end{aligned}
$$

Note that we derived the general solution

$$X_i(T_1) = -\frac{A_2n^2 + A_1n + A_0}{B_3n^3 + B_2n^2 + B_1n + B_0}X_0 \tag{60}$$

under the assumption that $n \geq 2$. Equation 60 is indeed not true for $n = 1$ in an *algebraic* sense, since both the numerator and denominator of Equation 60 are 0 in this case. However, the Equation 60 is *analytically* consistent with the case $n = 1$, since it converges for $n \to 1$ against the optimal value of $X_1(T_1)$ for $n = 1$ as given in Equation 12:

$$\begin{aligned}\lim_{n\to 1} X_1(T_1) &= -\lim_{n\to 1} \frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0} X_0 \\ &= -\frac{\left(-2 - e^{\frac{\gamma T_1}{3\lambda}} - e^{\frac{2\gamma T_1}{3\lambda}} + e^{\frac{\gamma T_1}{\lambda}}\right)\frac{\gamma}{\lambda}(T_2 - T_1)}{6\left(-1 + e^{\frac{\gamma T_1}{\lambda}}\right) + \left(2 + e^{\frac{\gamma T_1}{3\lambda}} + e^{\frac{2\gamma T_1}{3\lambda}} + 2e^{\frac{\gamma T_1}{\lambda}}\right)\frac{\gamma}{\lambda}(T_2 - T_1)} X_0 = X_1(T_1)_{n=1}. \quad (61)\end{aligned}$$

The necessity of two different algebraic expressions for the cases $n = 1$ and $n \geq 2$ therefore can be considered as mathematical, but not economic difficulties. In particular, they do not reflect an ill-behavedness of the model in the case $n = 1$.

In the following proofs, we will need the limits $\lim_{n\to\infty} A_i$ and $\lim_{n\to\infty} B_i$. All of these limits exist and can be established by direct calculations. We obtain:

$$\lim_{n\to\infty} A_0 = 2e^{\frac{\gamma T_1}{\lambda}}\Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)^2 \quad (62)$$

$$\lim_{n\to\infty} A_1 = -3\Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)^2 \quad (63)$$

$$\lim_{n\to\infty} A_2 = -\Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)^2 \quad (64)$$

$$\lim_{n\to\infty} B_0 = -2\Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)\Big(-1 + 2e^{\frac{\gamma T_1}{\lambda}} - 2e^{\frac{\gamma (T_2 - T_1)}{\lambda}} + e^{\frac{\gamma T_2}{\lambda}}\Big) \quad (65)$$

$$\lim_{n\to\infty} B_1 = \Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma T_2}{\lambda}}\Big) \quad (66)$$

$$\lim_{n\to\infty} B_2 = 4\Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma T_2}{\lambda}}\Big) \quad (67)$$

$$\lim_{n\to\infty} B_3 = \Big(-1 + e^{\frac{\gamma T_1}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma (T_2 - T_1)}{\lambda}}\Big)\Big(-1 + e^{\frac{\gamma T_2}{\lambda}}\Big) \quad (68)$$

(69)

□

Proof of Theorem 3. See Appendix C. □

Proof of Proposition 4. The calculation of Equations 14 and 15 is straightforward. □

Proof of Proposition 5. By Theorem 1, we know that

$$\dot{X}_i(t) = a e^{-\frac{\gamma}{\lambda} t} + b_1 e^{\frac{\gamma}{\lambda} t} \quad (70)$$

with

$$a = \frac{n}{(n+1)(n+2)} \frac{\gamma}{\lambda} \left(1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda} T_1}\right)^{-1} \left(\sum_{i=0}^{n} (X_i(T_1) - X_i(0))\right) < 0 \text{ by Proposition 4} \quad (71)$$

$$b_1 = \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda} T_1} - 1\right)^{-1} \left(X_1(T_1) - X_1(0) - \frac{\sum_{j=0}^{n} (X_j(T_1) - X_j(0))}{n+1}\right) > 0 \text{ by Proposition 4.} \quad (72)$$

The monotonicity of the trading rate of the competitors in time t now follows from Equation 70.

Both the initial trading rate $\dot{X}_i(0)$ for $\gamma T_1/\lambda = 0$ is a direct consequence of Equation 70 and Proposition 4. The monotonicity with respect to $\gamma T_1/\lambda$ follows from Equation 70 and Theorem 3. □

Proof of Corollary 6. This is a direct consequence of Theorem 3 and Proposition 5. □

Proof of Proposition 7. We apply Theorem 2 and obtain:

$$\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = -\frac{\lim_{n\to\infty} A_2}{\lim_{n\to\infty} B_3} X_0 \tag{73}$$

From the proof of Theorem 2, we know the values of the limits of A_2 and B_3 and the desired result follows. □

Proof of Corollary 8. Using Proposition 7 and L'Hospitale's rule, we calculate

$$\lim_{\lambda\to\infty}\lim_{n\to\infty}\sum_{i=1}^{n} X_i(T_1) = \lim_{\lambda\to\infty}\frac{e^{\frac{\gamma(T_2-T_1)}{\lambda}}-1}{e^{\frac{\gamma T_2}{\lambda}}-1} = \frac{T_2-T_1}{T_2} \tag{74}$$

□

Proof of Corollary 9. We observe that by Theorem 2 all derivatives of $X_i(T_1)$ converge locally uniformly. Hence, we have

$$\lim_{n\to\infty}\frac{d}{d\gamma}X_i(T_1) = \frac{d}{d\gamma}\lim_{n\to\infty} X_i(T_1) \tag{75}$$

and by computing the derivatives of $\lim_{n\to\infty} X_i(T_1)$ using Proposition 7 we obtain the first two relations of the corollary. Similar to the proof of Theorem 11, it can be shown that for large n, $X_i(T_1)$ is increasing in n. This shows the last of the three relations stated in the corollary.

□

Proof of Proposition 10. By Theorem 1, we know that

$$\lim_{n\to\infty}\sum_{i=1}^{n}\dot{X}_i(t) = \left(\lim_{n\to\infty} na\right)e^{-\frac{\gamma}{\lambda}t} + \left(\lim_{n\to\infty} nb_1\right)e^{\frac{\gamma}{\lambda}t} \tag{76}$$

with

$$\begin{aligned}
\lim_{n\to\infty} na &= \frac{\gamma}{\lambda}\left(1-e^{-\frac{\gamma}{\lambda}T_1}\right)^{-1}\left(\sum_{i=0}^{n}(X_i(T_1)-X_i(0))\right) && (77)\\
&= \frac{\gamma}{\lambda}\left(1-e^{-\frac{\gamma}{\lambda}T_1}\right)^{-1}\frac{e^{\frac{\gamma}{\lambda}(T_2-T_1)}-e^{\frac{\gamma}{\lambda}T_2}}{e^{\frac{\gamma}{\lambda}T_2}-1}X_0 && (78)\\
\lim_{n\to\infty} nb_1 &= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1}-1\right)^{-1}\left(n(X_1(T_1)-X_1(0))-\left(\sum_{j=0}^{n}(X_j(T_1)-X_j(0))\right)\right) && (79)\\
&= \frac{\gamma}{\lambda}\left(e^{\frac{\gamma}{\lambda}T_1}-1\right)^{-1}X_0. && (80)
\end{aligned}$$

Hence we obtain

$$\lim_{n\to\infty}\sum_{i=1}^{n}\dot{X}_i(0) = \left(\lim_{n\to\infty} na\right) + \left(\lim_{n\to\infty} nb_1\right) \tag{81}$$

$$= \frac{\gamma}{\lambda}\frac{2e^{\frac{\gamma}{\lambda}T_2} - e^{\frac{\gamma}{\lambda}(T_1+T_2)} - 1}{\left(e^{\frac{\gamma}{\lambda}T_1} - 1\right)\left(e^{\frac{\gamma}{\lambda}T_2} - 1\right)}X_0. \tag{82}$$

Expression 24 is positive if and only if

$$2e^{\frac{\gamma}{\lambda}T_2} - e^{\frac{\gamma}{\lambda}(T_1+T_2)} - 1 > 0 \tag{83}$$

$$\Leftrightarrow \qquad \frac{T_2}{T_1} > -\frac{\ln\left(2 - e^{\frac{\gamma}{\lambda}T_1}\right)}{\frac{\gamma}{\lambda}T_1}. \tag{84}$$

□

Proof of Theorem 11. Using Theorems 1 and 2 and Propositions A.1 and 13, we can calculate the return for the seller in a straightforward way and obtain:

$$R_0 = X_0\left(P_0 - \gamma X_0\frac{A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0}{B_7n^7 + B_6n^6 + B_5n^5 + B_4n^4 + B_3n^3 + B_2n^2 + B_1n + B_0}\right) \tag{85}$$

That is, we can set

$$G\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) := \frac{A_7n^7 + A_6n^6 + A_5n^5 + A_4n^4 + A_3n^3 + A_2n^2 + A_1n + A_0}{B_7n^7 + B_6n^6 + B_5n^5 + B_4n^4 + B_3n^3 + B_2n^2 + B_1n + B_0}.$$

The coefficients A_i and B_i are functions of $\frac{\gamma T_1}{\lambda}$, $\frac{T_2}{T_1}$ and n. They are of a similar structure as the coefficients derived in the proof of Theorem 2, but even more complex. The calculations and coefficients are omitted here for brevity (they are available from the authors on request).

The coefficients A_i and B_i converge for $n \to \infty$; furthermore, their derivatives $\frac{dA_i}{dn}$ and $\frac{dB_i}{dn}$ converge to 0 as $n \to \infty$. We compute

$$\lim_{n\to\infty} R_0 = \lim_{n\to\infty}\mathbb{E}(\text{Return for the seller}) = X_0\left(P_0 - \gamma X_0\frac{\lim_{n\to\infty} A_7}{\lim_{n\to\infty} B_7}\right). \tag{86}$$

Inserting A_7 and B_7 and computing the limit gives the desired limit.

To prove that $\lim_{n\to\infty} R_0$ is increasing for large n, we compute the derivative of the seller's return R_0 with respect to n as

$$\frac{d}{dn}R_0 = -\gamma X_0\frac{\text{Numerator}}{(B_7n^7 + B_6n^6 + B_5n^5 + B_4n^4 + B_3n^3 + B_2n^2 + B_1n + B_0)^2} \tag{87}$$

with

$$\begin{aligned}\text{Numerator} = \Bigg(&7A_7B_7n + 7A_7B_6 + 6A_6B_7 + \frac{dA_7}{dn}B_7n^2 + \frac{dA_7}{dn}B_6n \\ &+ \frac{dA_7}{dn}B_5 + \frac{dA_6}{dn}B_7n + \frac{dA_6}{dn}B_6 + \frac{dA_5}{dn}B_7\Bigg)n^{12} \\ &- \Bigg(7B_7A_7n + 7B_7A_6 + 6B_6A_7 + \frac{dB_7}{dn}A_7n^2 + \frac{dB_7}{dn}A_6n \\ &+ \frac{dB_7}{dn}A_5 + \frac{dB_6}{dn}A_7n + \frac{dB_6}{dn}A_6 + \frac{dB_5}{dn}A_7\Bigg)n^{12} + o(n^{11}).\end{aligned} \tag{88}$$

For large n, we can omit the $o(n^{11})$ term; furthermore, we know that all derivatives converge to 0 as $n \to \infty$. We therefore obtain for large n:

$$\begin{aligned}\text{Numerator} \approx \Bigg(&\left(\frac{dA_7}{dn}B_7 - \frac{dB_7}{dn}A_7\right) n^2 \\ &+ \left(\frac{dA_7}{dn}B_6 + \frac{dA_6}{dn}B_7 - \frac{dB_7}{dn}A_6 n - \frac{dB_6}{dn}A_7\right) n \\ &+ A_7 B_6 - B_7 A_6 \Bigg) n^{12} \end{aligned} \tag{89}$$

Inserting the expressions for A_i and B_i, we obtain

$$\lim_{n\to\infty} \left(\frac{dA_7}{dn}B_7 - \frac{dB_7}{dn}A_7\right) n^2 = 0 \tag{90}$$

$$\lim_{n\to\infty} \left(\frac{dA_7}{dn}B_6 + \frac{dA_6}{dn}B_7 - \frac{dB_7}{dn}A_6 n - \frac{dB_6}{dn}A_7\right) n = 0 \tag{91}$$

$$\lim_{n\to\infty} (A_7B_6 - B_7A_6) = -e^{\frac{\gamma T_1}{\lambda}}\left(e^{\frac{\gamma T_1}{\lambda}} - 1\right)^7 \left(e^{\frac{\gamma(T_2-T_1)}{\lambda}} - 1\right)^5 \left(e^{\frac{\gamma T_2}{\lambda}} - 1\right)^3 < 0. \tag{92}$$

The derivative of the seller's return has the opposite sign of the $Numerator$ and is thus positive for large values of n.

To prove that the seller's return is decreasing in $\gamma T_1/\lambda$ and increasing in T_2/T_1 for large n, we proceed similar to the proof of Corollary 9, observe that the derivatives of R_0 converge locally uniformly for $n \to \infty$ and obtain the desired relations by inspection of the limit $\lim_{n\to\infty} R_0$. □

Proof of Proposition 12. The condition

$$\frac{1}{2} + \frac{\lambda}{\gamma T_1} > \frac{1}{1 - e^{-\frac{\gamma}{\lambda}T_2}}. \tag{93}$$

is obtained by direct comparison of the returns of sunshine and stealth trading given in Equations 30 and 31. Equation 33 can be derived by passing to the limit $T_2 \to \infty$. □

Proof of Proposition 13. First, we note that by arguments similar to the proof of Proposition A.1 (in particular Formula (48)), the price during stage 1 ($t \in [0, T_1)$) is

$$\begin{aligned}\bar{P}(t) = P_0 &- \gamma \left(X_0 - \sum_{i=1}^n X_i(T_1)\right) \frac{1}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}} \\ &+ \gamma \left(X_0 - \sum_{i=1}^n X_i(T_1)\right) \frac{2}{n+2} \frac{e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}t}}{1 - e^{-\frac{n}{n+2}\frac{\gamma}{\lambda}T_1}}\end{aligned} \tag{94}$$

and the price during stage 2 ($t \in [T_1, T_2]$) is

$$\begin{aligned}\bar{P}(t) = P_0 &- \gamma \left(X_0 - \sum_{i=1}^n X_i(T_1)\right) - \gamma \left(\sum_{i=1}^n X_i(T_1)\right) \frac{1}{1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2-T_1)}} \\ &+ \gamma \left(\sum_{i=1}^n X_i(T_1)\right) \frac{2}{n+1} \frac{e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(t-T_1)}}{1 - e^{-\frac{n-1}{n+1}\frac{\gamma}{\lambda}(T_2-T_1)}}.\end{aligned} \tag{95}$$

Again, the time-dependent terms vanish as n increases. For the first stage, we obtain the limit

$$
\begin{aligned}
\lim_{n\to\infty} \bar{P}(t) &= P_0 - \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n\to\infty} \frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} \\
&\quad + \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \left(\lim_{n\to\infty} \frac{2}{n+2} \right) \frac{e^{-\lim_{n\to\infty} \frac{n}{n+2} \frac{\gamma}{\lambda} t}}{1 - e^{-\lim_{n\to\infty} \frac{n}{n+2} \frac{\gamma}{\lambda} T_1}} && (96) \\
&= P_0 - \gamma \left(X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda} T_1}} && (97) \\
&= P_0 - \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1} && (98)
\end{aligned}
$$

For the second stage, we compute

$$
\begin{aligned}
\lim_{n\to\infty} \bar{P}(t) &= P_0 - \gamma \left(X_0 - \lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \\
&\quad - \gamma \left(\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \frac{1}{1 - e^{-\lim_{n\to\infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}} \\
&\quad + \gamma \left(\lim_{n\to\infty} \sum_{i=1}^{n} X_i(T_1) \right) \lim_{n\to\infty} \frac{2}{n+1} \frac{e^{-\lim_{n\to\infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (t - T_1)}}{1 - e^{-\lim_{n\to\infty} \frac{n-1}{n+1} \frac{\gamma}{\lambda} (T_2 - T_1)}} && (99) \\
&= P_0 - \gamma \left(X_0 - \frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \\
&\quad - \gamma \left(\frac{e^{\frac{\gamma(T_2 - T_1)}{\lambda}} - 1}{e^{\frac{\gamma T_2}{\lambda}} - 1} X_0 \right) \frac{1}{1 - e^{-\frac{\gamma}{\lambda} (T_2 - T_1)}} && (100) \\
&= P_0 - \gamma X_0 \frac{e^{\frac{\gamma T_2}{\lambda}}}{e^{\frac{\gamma T_2}{\lambda}} - 1}. && (101)
\end{aligned}
$$

□

Proof of Proposition 14. By Formulas 94 and 95, it is easy to see that within each stage the price $\bar{P}(t)$ moves monotonously. Therefore, the only four possible times at which the minimum price can be achieved are T_0, T_1-, T_1 and T_2. It is straightforward to calculate the prices for these four points in time using Theorem 2 and Formulas 94 and 95, to show that $\bar{P}(T_1-)$ is the minimum of these four values and that it is lower than $\bar{P}(T_2+)$. Furthermore, it is direct to show that $\bar{P}(T_1-)$ is an increasing function of the number of competitors n.

The different effect of competitors on price overshooting in plastic and elastic markets is shown by the examples in Section IV. □

C Proof of Theorem 3

In the following, we outline the proof for $n \geq 3$; the proofs for the cases $n = 1$ and $n = 2$ are identical in approach but simpler.

Recall from Theorem 2 and Equation 59 that

$$X_i(T_1)/X_0 = F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) = -\frac{A_2 n^2 + A_1 n + A_0}{B_3 n^3 + B_2 n^2 + B_1 n + B_0}. \tag{102}$$

In the following, we show that the amount of liquidity provision falls in $\gamma T_1/\lambda$ for all $n \geq 1$:

$$\frac{\partial}{\partial(\gamma T_1/\lambda)} F\left(\frac{\gamma T_1}{\lambda}, \frac{T_2}{T_1}, n\right) \leq 0 \quad \text{for all } \frac{\gamma T_1}{\lambda} \geq 0, \frac{T_2}{T_1} \geq 1, n \geq 3 \tag{103}$$

Without loss of generality, we will assume that $\lambda = 1$, $T_1 = n + 2$ and set

$$T := (T_2 - T_1)/(n+1) = (T_2 - n - 2)/(n+1) \tag{104}$$

for the rest of this proof. Then Equation 103 is equivalent to

$$\begin{aligned} C(\gamma, T, n) := -\Big[&\left(\frac{\partial}{\partial\gamma}\left(A_2 n^2 + A_1 n + A_0\right)\right)\left(B_3 n^3 + B_2 n^2 + B_1 n + B_0\right) \\ &- \left(A_2 n^2 + A_1 n + A_0\right)\left(\frac{\partial}{\partial\gamma}\left(B_3 n^3 + B_2 n^2 + B_1 n + B_0\right)\right)\Big]/(n+2) < 0 \\ &\text{for all } \gamma \geq 0, T \geq 0, n \geq 3. \end{aligned} \tag{105}$$

Based on the formulas provided in the proof of Theorem 2, we can express the function C as

$$C(\gamma, T, n) = \sum_{a=1}^{9}\sum_{b=1}^{9}(\widehat{\text{Coeff}}_{a,b}(n) + \widetilde{\text{Coeff}}_{a,b}(n)T)e^{\gamma(p_a(n) + \widetilde{p}_b(n)T)} \tag{106}$$

where $\widehat{\text{Coeff}}_{a,b}(n), \widetilde{\text{Coeff}}_{a,b}(n), p_a(n)$ and $\widetilde{p}_b(n)$ are polynomials in n:

$$p_1(n) = 0 \qquad p_2(n) = n \qquad p_3(n) = n + 2 \tag{107}$$
$$p_4(n) = 2n \qquad p_5(n) = 2n + 2 \qquad p_6(n) = 2n + 4 \tag{108}$$
$$p_7(n) = 3n + 2 \qquad p_8(n) = 3n + 4 \qquad p_9(n) = 4n + 4 \tag{109}$$

$$\widetilde{p}_1(n) = 2 \qquad \widetilde{p}_2(n) = n + 1 \qquad \widetilde{p}_3(n) = n + 3 \tag{110}$$
$$\widetilde{p}_4(n) = 2n \qquad \widetilde{p}_5(n) = 2n + 2 \qquad \widetilde{p}_6(n) = 2n + 4 \tag{111}$$
$$\widetilde{p}_7(n) = 3n + 1 \qquad \widetilde{p}_8(n) = 3n + 3 \qquad \widetilde{p}_9(n) = 4n + 2 \tag{112}$$

See Table III for a list of $\widehat{\text{Coeff}}_{a,b}(n)$ and $\widetilde{\text{Coeff}}_{a,b}(n)$.

We can write $C(\gamma, T, n)$ as a power series in γ and T:

$$C(\gamma, T, n) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}\frac{\partial^{i+j}C}{\partial\gamma^i\partial T^j}(0,0,n)\gamma^i T^j \tag{113}$$

We will show that

$$\frac{\partial^{i+j}C}{\partial\gamma^i\partial T^j}(0,0,n) \leq 0 \text{ for all } i,j \geq 0 \text{ and } n \geq 3. \tag{114}$$

$a\downarrow b\rightarrow$	1	2	3	4	5	6	7	8	9
1									
2	$2n^3-2n^2$		$2n^4-4n^3+2n^2$	$-2n^3+2n^2$	$-4n^4+4n^2$	$-2n^4+2n^3$	$2n^4+4n^3-6n^2$	$4n^4-4n^2$	$-2n^4-2n^3+4n^2$
3	$-4n^3-14n^2-10n+4$	$2n^4+12n^3+22n^2+12n$	$n^5+3n^4+7n^3+21n^2+20n-4$	$-2n^4-8n^3-8n^2-2n-4$	$-2n^5-10n^4-22n^3-30n^2-24n-8$	$-n^5-3n^4-3n^3-7n^2-10n$	$n^5+7n^4+15n^3+9n^2+4n+12$	$2n^5+8n^4+10n^3+8n^2+12n+8$	$-n^5-5n^4-7n^3-n^2-2n-8$
4									
5	$4n^3+28n^2+16n-8$	$-4n^4-24n^3-36n^2-16n$	$-2n^5-10n^4-6n^3-38n^2-32n+8$	$4n^4+20n^3+8n^2+8$	$4n^5+28n^4+44n^3+36n^2+32n+16$	$2n^5+10n^4+2n^3+10n^2+16n$	$-2n^5-18n^4-38n^3+2n^2-24$	$-4n^5-24n^4-20n^3-16n-16$	$2n^5+14n^4+18n^3-10n^2+16$
6									
7	$-2n^3-8n^2-6n+4$	$2n^4+8n^3+10n^2+4n$	$n^5+3n^4+3n^3+9n^2+12n-4$	$-2n^4-6n^3-2n^2+2n-4$	$-2n^5-10n^4-14n^3-6n^2-8n-8$	$-n^5-3n^4-n^3-n^2-6n$	$n^5+7n^4+11n^3-3n^2-4n+12$	$2n^5+8n^4+6n^3-4n^2+4n+8$	$-n^5-5n^4-5n^3+5n^2+2n-8$
8	$-4n^2$	$4n^3+4n^2$	$2n^4+6n^2$	$-4n^3$	$-4n^4-8n^3-4n^2$	$-2n^4-2n^2$	$2n^4+8n^3-2n^2$	$4n^4+4n^3$	$-2n^4-4n^3+2n^2$
9									

$a\downarrow b\rightarrow$	1	2	3	4	5	6	7	8	9
1		$-2n^4-2n^3+10n^3-6n$	$-n^5-4n^4-2n^3+4n^2+3n$		$2n^5+12n^4+8n^3-28n^2+6n$		$-n^5-4n^4-2n^3+4n^2+3n$	$-2n^4-2n^3+10n^2-6n$	
2		$4n^4-12n^2+8n$	$2n^5+6n^4+2n^3-6n^2-4n$		$-4n^5-20n^4-4n^3+36n^2-8n$		$2n^5+6n^4+2n^3-6n^2-4n$	$4n^4-12n^2+8n$	
3		$2n^4+6n^3-14n^2+2n+4$	$n^5+6n^4+6n^3-4n^2-7n-2$		$-2n^5-16n^4-24n^3+36n^2+10n-4$		$n^5+6n^4+6n^3-4n^2-7n-2$	$2n^4+6n^3-14n^2+2n+4$	
4		$-2n^4+2n^3+2n^2-2n$	$-n^5-2n^4+2n^2+n$		$2n^5+8n^4-4n^3-8n^2+2n$		$-n^5-2n^4+2n^2+n$	$-2n^4+2n^3+2n^2-2n$	
5		$-4n^4-4n^3+12n^2+4n-8$	$-2n^5-8n^4-8n^3+4n^2+10n+4$		$4n^5+24n^4+24n^3-32n^2-28n+8$		$-2n^5-8n^4-8n^3+4n^2+10n+4$	$-4n^4-4n^3+12n^2+4n-8$	
6		$-4n^3+4n^2+4n-4$	$-2n^4-4n^3+4n+2$		$4n^4+16n^3-8n^2-16n+4$		$-2n^4-4n^3+4n+2$	$-4n^3+4n^2+4n-4$	
7		$2n^4-2n^3+2n^2-6n+4$	$n^5+2n^4+2n^3-3n-2$		$-2n^5-8n^4-4n^2+18n-4$		$n^5+2n^4+2n^3-3n-2$	$2n^4-2n^3+2n^2-6n+4$	
8		$4n^3-12n+8$	$2n^4+6n^3+2n^2-6n-4$		$-4n^4-20n^3-4n^2+36n-8$		$2n^4+6n^3+2n^2-6n-4$	$4n^3-12n+8$	
9		$-4n^2+8n-4$	$-2n^3-2n^2+2n+2$		$4n^3+12n^2-20n+4$		$-2n^3-2n^2+2n+2$	$-4n^2+8n-4$	

Table III: Coeffcients $\widehat{\mathrm{Coeff}}_{a,b}(n)$ (top table) and $\widetilde{\mathrm{Coeff}}_{a,b}(n)$ (bottom table). The cell in row a and column b shows $\widehat{\mathrm{Coeff}}_{a,b}(n)$ resp. $\widetilde{\mathrm{Coeff}}_{a,b}(n)$.

Then Equation 105 and thus the theorem follow directly from Equation 113. Given our expression of C in Equation 106, we can infer that

$$\frac{\partial^{i+j}C}{\partial\gamma^i\partial T^j}(0,0,n)=\begin{cases} i\geq j\geq 1: & \sum_{a,b=1}^{9}\Big(\widehat{\mathrm{Coeff}}_{a,b}(n)p_a(n)^{i-j}\widetilde{p}_b(n)^j\binom{i}{j}j! \\ & \quad +\widetilde{\mathrm{Coeff}}_{a,b}(n)p_a(n)^{i-j+1}\widetilde{p}_b(n)^{j-1}\binom{i}{j-1}j!\Big) \\ i=j-1: & \sum_{a,b=1}^{9}\widetilde{\mathrm{Coeff}}_{a,b}(n)\widetilde{p}_b(n)^{j-1}j! \\ i\leq j-2: & 0 \\ j=0: & \sum_{a,b=1}^{9}\widehat{\mathrm{Coeff}}_{a,b}(n)p_a(n)^i. \end{cases} \tag{115}$$

A direct calculation reveals that for all $1\leq a\leq 9$

$$\sum_b\widehat{\mathrm{Coeff}}_{a,b}(n)=0 \qquad \sum_b\widetilde{\mathrm{Coeff}}_{a,b}(n)=0 \tag{116}$$

$$\sum_b\widehat{\mathrm{Coeff}}_{a,b}(n)\widetilde{p}_b(n)=0 \qquad \sum_b\widetilde{\mathrm{Coeff}}_{a,b}(n)\widetilde{p}_b(n)=0 \tag{117}$$

$$\sum_b\widehat{\mathrm{Coeff}}_{a,b}(n)\widetilde{p}_b(n)^2=0 \tag{118}$$

and for all $1\leq b\leq 9$

$$\sum_a\widehat{\mathrm{Coeff}}_{a,b}(n)=0 \qquad \sum_a\widetilde{\mathrm{Coeff}}_{a,b}(n)=0 \tag{119}$$

$$\sum_a\widehat{\mathrm{Coeff}}_{a,b}(n)p_a(n)=0 \qquad \sum_a\widetilde{\mathrm{Coeff}}_{a,b}(n)p_a(n)=0 \tag{120}$$

$$\sum_a\widetilde{\mathrm{Coeff}}_{a,b}(n)p_a(n)^2=0. \tag{121}$$

We can therefore conclude that

$$\frac{\partial^{i+j}C}{\partial\gamma^i\partial T^j}(0,0,n)=0 \tag{122}$$

for $i=j-1$, $i=j$, $i=j+1$, $j=0$, $j=1$ and $j=2$. We thus only need to show that

$$\mathrm{Pow}_{i,j}(n):=\sum_{a,b=1}^{9}\Big(\widehat{\mathrm{Coeff}}_{a,b}(n)p_a(n)^{i-j}\widetilde{p}_b(n)^j(i-j+1)+\widetilde{\mathrm{Coeff}}_{a,b}(n)p_a(n)^{i-j+1}\widetilde{p}_b(n)^{j-1}j\Big)$$
$$\leq 0 \text{ for all } j\geq 3, i\geq j+2. \tag{123}$$

Each $\mathrm{Pow}_{i,j}$ is again a polynomial in n. We now show that these polynomials attain only nonpositive values. We divide the proof into the following four lemmas:

Lemma 15. *The expression $Pow_{i,j}(n)$ is nonpositive if $29\leq j$, $j+2\leq i$ and $3\leq n$.*

Lemma 16. *The expression $Pow_{i,j}(n)$ is nonpositive if $3\leq j\leq 28$, $j+33\leq i$ and $3\leq n\leq 35$.*

Lemma 17. *The expression $Pow_{i,j}(n)$ is nonpositive if $3\leq j\leq 28$, $4j\leq i$ and $36\leq n$.*

Lemma 18. *The expression $Pow_{i,j}(n)$ is nonpositive if $3\leq j\leq 28$, $j+2\leq i\leq max(j+32,4j-1)$ and $3\leq n$.*

For the proofs of these lemmas, we decompose $\mathrm{Pow}_{i,j}(n)$ into six terms:

$$S_A := \{(a,b) \in \mathbb{N}^2 | (a \le 8 \text{ and } b \le 6) \text{ or } (a \le 6 \text{ and } b \le 8)\} \tag{124}$$
$$S_B := \{(a,b) \in \mathbb{N}^2 | 7 \le a \le 8 \text{ and } 7 \le b \le 8\} \tag{125}$$
$$S_C := \{(a,b) \in \mathbb{N}^2 | 1 \le a \le 8 \text{ and } b = 9\} \tag{126}$$
$$S_D := \{(a,b) \in \mathbb{N}^2 | a = 9 \text{ and } 1 \le b \le 9\} \tag{127}$$
$$A \cup B \cup C \cup D = \{(a,b) \in \mathbb{N}^2 | 1 \le a \le 9 \text{ and } 1 \le b \le 9\} \tag{128}$$

$$\widehat{A}_{i,j}(n) := \sum_{(a,b)\in S_A} \widehat{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j} \widetilde{p}_b(n)^j (i-j+1) \tag{129}$$
$$\widetilde{A}_{i,j}(n) := \sum_{(a,b)\in S_A} \widetilde{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j+1} \widetilde{p}_b(n)^{j-1} j \tag{130}$$
$$\widehat{B}_{i,j}(n) := \sum_{(a,b)\in S_B} \widehat{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j} \widetilde{p}_b(n)^j (i-j+1) \tag{131}$$
$$\widetilde{B}_{i,j}(n) := \sum_{(a,b)\in S_B} \widetilde{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j+1} \widetilde{p}_b(n)^{j-1} j \tag{132}$$
$$\widehat{C}_{i,j}(n) := \sum_{(a,b)\in S_C} \widehat{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j} \widetilde{p}_b(n)^j (i-j+1) \tag{133}$$
$$\widetilde{D}_{i,j}(n) := \sum_{(a,b)\in S_D} \widetilde{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j+1} \widetilde{p}_b(n)^{j-1} j \tag{134}$$
$$\mathrm{Pow}_{i,j}(n) = \widehat{A}_{i,j}(n) + \widetilde{A}_{i,j}(n) + \widehat{B}_{i,j}(n) + \widetilde{B}_{i,j}(n) + \widehat{C}_{i,j}(n) + \widetilde{D}_{i,j}(n) \tag{135}$$

Before proving Lemmas 15 until 18, we state and prove two auxiliary lemmas.

Auxiliary Lemma 19. *If $3 \le j$ and $j+2 \le i$, then $\widehat{A}_{i,j} \le 0$, $\widetilde{A}_{i,j} \le 0$, $\widehat{C}_{i,j} \le 0$ and $\widetilde{D}_{i,j} \le 0$ for all $n \ge 3$.*

Proof of Auxiliary Lemma 19. We first show that $\widetilde{D}_{i,j} \le 0$. This follows directly by Jensen's inequality since x^{j-1} is convex:

$$\widetilde{D}_{i,j}(n) = \sum_{(a,b)\in S_D} \widetilde{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j+1} \widetilde{p}_b(n)^{j-1} j \tag{136}$$
$$= p_9(n)^{i-j+1} j \sum_{b=1}^{9} \widetilde{\mathrm{Coeff}}_{9,b}(n) \widetilde{p}_b(n)^{j-1} \tag{137}$$
$$= \underbrace{p_9(n)^{i-j+1} j}_{\ge 0} \Big(\underbrace{(-2n^3 - 2n^2 + 2n + 2)}_{\le 0} \underbrace{((3n+1)^{j-1} - 2(2n+2)^{j-1} + (n+3)^{j-1})}_{\ge 0 \text{ (Jensen)}} \tag{138}$$
$$+ \underbrace{(-4n^2 + 8n - 4)}_{\le 0} \underbrace{((3n+3)^{j-1} - 2(2n+2)^{j-1} + (n+1)^{j-1})}_{\ge 0 \text{ (Jensen)}} \Big)$$

We now show that $\widehat{C}_{i,j} \leq 0$. First, we apply Jensen again to reduce $\widehat{C}_{i,j}$:

$$
\begin{aligned}
\widehat{C}_{i,j}(n) =& (i-j+1)\widetilde{p}_9(n)^j \sum_{a=1}^{8} \widehat{\mathrm{Coeff}}_{a,9}(n) p_a(n)^{i-j} && (139)\\
=& (i-j+1)\widetilde{p}_9(n)^j && (140)\\
& \Big((-n^5 - 5n^4 - 5n^3 + 5n^2 + 2n - 8)\big((3n+2)^{i-j} - 2(2n+2)^{i-j} + (n+2)^{i-j}\big)\\
& + (-2n^4 - 2n^3 + 4n^2)\big((3n+4)^{i-j} - 2(2n+2)^{i-j} + n^{i-j}\big)\\
& + (-2n^3 - 2n^2)(3n+4)^{i-j}\\
& + (4n^3 + 8n^2 + 4n)(2n+2)^{i-j} + (-2n^3 - 6n^2 - 4n)(n+2)^{i-j}\Big)\\
\overset{\text{Jensen}}{\leq}& (i-j+1)\widetilde{p}_9(n)^j \Big((-2n^3 - 2n^2)(3n+4)^{i-j} && (141)\\
& + (4n^3 + 8n^2 + 4n)(2n+2)^{i-j} + (-2n^3 - 6n^2 - 4n)(n+2)^{i-j}\Big)\\
\leq& (i-j+1)\widetilde{p}_9(n)^j (2n+2)^{i-j} \Big((-2n^3 - 2n^2)(3/2)^{i-j} && (142)\\
& + (4n^3 + 8n^2 + 4n) + (-2n^3 - 6n^2 - 4n)(1/2)^{i-j}\Big)
\end{aligned}
$$

By the conditions stated in the lemma, we have $j+2 \leq i$. We differentiate two cases: For $i = j+2$, we have

$$
\begin{aligned}
\widehat{C}_{j+2,j}(n) \leq& 3\widetilde{p}_9(n)^j (2n+2)^2 \Big((-2n^3 - 2n^2)(9/4) && (143)\\
& + (4n^3 + 8n^2 + 4n) + (-2n^3 - 6n^2 - 4n)(1/4)\Big)\\
=& \underbrace{3\widetilde{p}_9(n)^j (2n+2)^2}_{\geq 0} \underbrace{(-n^3 + 2n^2 + 3n)}_{\leq 0 \text{ since } n \geq 3} \leq 0. && (144)
\end{aligned}
$$

For $i \geq j+3$, we have

$$
\begin{aligned}
\widehat{C}_{i,j}(n) \leq& (i-j+1)\widetilde{p}_9(n)^j (2n+2)^{i-j} \Big((-2n^3 - 2n^2)(27/8) + (4n^3 + 8n^2 + 4n)\Big) && (145)\\
=& \underbrace{(i-j+1)\widetilde{p}_9(n)^j (2n+2)^{i-j}}_{\geq 0} \underbrace{(-(11/4)n^3 + (5/4)n^2 + 4n)}_{\leq 0 \text{ since } n \geq 3} \leq 0. && (146)
\end{aligned}
$$

We conclude by showing that $\widehat{A}_{i,j}, \widetilde{A}_{i,j} \leq 0$. We define

$$
\widehat{A}^{(k)}_{i,j}(n) := \sum_{(a,b) \in S_A \text{ and } b \geq k} \widehat{\mathrm{Coeff}}_{a,b}(n) p_a(n)^{i-j}. \tag{147}
$$

By explicit calculation of the $\widehat{A}^{(k)}_{i,j}$ for $1 \leq k \leq 8$, we see that these are all nonpositive. It follows that $\widehat{A}_{i,j} \leq 0$.

$\widetilde{A}_{i,j} \leq 0$ follows by a similar argument after an application of Jensen's inequality. $\square$

Auxiliary Lemma 20. *The expression $\widehat{A}_{i,j}(n) + \widehat{B}_{i,j}(n) + \widehat{C}_{i,j}(n)$ is nonpositive for all $j \geq 3$, $i \geq j+2$ and $n \geq 3$.*

Proof. By explicit computation, we obtain

$$\widehat{A}_{i,j}(n)+\widehat{B}_{i,j}(n)+\widehat{C}_{i,j}(n)=\sum_{a,b=1}^{9}\widehat{\mathrm{Coeff}}_{a,b}(n)p_a(n)^{i-j}\widetilde{p}_b(n)^j(i-j+1) \tag{148}$$

$$=(i-j+1)\sum_{k=0}^{5}P_kQ_k \tag{149}$$

where

$$P_0 :=(-2n^3-2n^2)(3n+4)^{i-j}+(4n^3+8n^2+4n)(2n+2)^{i-j} \tag{150}$$
$$+(-2n^3-6n^2-4n)(n+2)^{i-j} \tag{151}$$
$$Q_0 :=\widetilde{p}_9(n)^j-2\widetilde{p}_8(n)^j-2\widetilde{p}_7(n)^j+\widetilde{p}_6(n)^j+4\widetilde{p}_5(n)^j \tag{152}$$
$$+\widetilde{p}_4(n)^j-2\widetilde{p}_3(n)^j-2\widetilde{p}_2(n)^j+\widetilde{p}_1(n)^j \tag{153}$$
$$P_1 :=-2n^3+2n^2 \tag{154}$$
$$Q_1 :=(p_8(n)^{i-j}-2p_5(n)^{i-j}+p_2(n)^{i-j}) \tag{155}$$
$$(\widetilde{p}_9(n)^j-2\widetilde{p}_7(n)^j-\widetilde{p}_6(n)^j+\widetilde{p}_4(n)^j+2\widetilde{p}_3(n)^j-\widetilde{p}_1(n)^j) \tag{156}$$
$$P_2 :=-2n^4+2n^2 \tag{157}$$
$$Q_2 :=(p_8(n)^{i-j}-2p_5(n)^{i-j}+p_2(n)^{i-j}) \tag{158}$$
$$(\widetilde{p}_9(n)^j-2\widetilde{p}_8(n)^j-\widetilde{p}_7(n)^j+\widetilde{p}_6(n)^j+2\widetilde{p}_5(n)^j-\widetilde{p}_3(n)^j) \tag{159}$$
$$P_3 :=-2n^3-8n^2-6n+4 \tag{160}$$
$$Q_3 :=(p_7(n)^{i-j}-2p_5(n)^{i-j}+p_3(n)^{i-j})Q_0 \tag{161}$$
$$P_4 :=-2n^4-4n^3+6n^2+8n-8 \tag{162}$$
$$Q_4 :=(p_7(n)^{i-j}-2p_5(n)^{i-j}+p_3(n)^{i-j}) \tag{163}$$
$$(\widetilde{p}_9(n)^j-\widetilde{p}_8(n)^j-2\widetilde{p}_7(n)^j+2\widetilde{p}_5(n)^j+\widetilde{p}_4(n)^j-\widetilde{p}_2(n)^j) \tag{164}$$
$$P_5 :=-n^5-3n^4+n^3+7n^2-4 \tag{165}$$
$$Q_5 :=(p_7(n)^{i-j}-2p_5(n)^{i-j}+p_3(n)^{i-j}) \tag{166}$$
$$(\widetilde{p}_9(n)^j-2\widetilde{p}_8(n)^j-\widetilde{p}_7(n)^j+\widetilde{p}_6(n)^j+2\widetilde{p}_5(n)^j-\widetilde{p}_3(n)^j). \tag{167}$$

It is easy to see that $P_k \leq 0$ for all k. $Q_k \geq 0$ follows from a generalized Jensen inequality for all k. $\square$

Proof of Lemma 15. We show that if the conditions of the lemma are met, then

$$\widehat{B}_{i,j}(n)+\widetilde{B}_{i,j}(n)+\widehat{C}_{i,j}(n)\leq 0. \tag{168}$$

By Equation 141 and the subsequent proof of $\widehat{C}_{i,j}(n) \le 0$, we know (since $n \ge 3$ and $i \ge j+2$):

$$\begin{aligned}
\widehat{C}_{i,j}(n) \le & (-n^5 - 5n^4 - 5n^3 + 5n^2 + 2n - 8)\big((3n+2)^{i-j} - 2(2n+2)^{i-j} + (n+2)^{i-j}\big)(4n+2)^j(i-j+1) \\
& + (-2n^4 - 2n^3 + 4n^2)\big((3n+4)^{i-j} - 2(2n+2)^{i-j} + n^{i-j}\big)(4n+2)^j(i-j+1) \qquad (169) \\
\le & (-n^5 - 5n^4 - 5n^3 + 5n^2 + 2n - 8)(3n+2)^{i-j}(4n+2)^j(i-j+1)\big(1 - 2(8/11)^2 + (5/11)^2\big) \\
& + (-2n^4 - 2n^3 + 4n^2)(3n+4)^{i-j}(4n+2)^j(i-j+1)\big(1 - 2(2/3)^2 + (1/3)^2\big) \qquad (170) \\
= & (18/121)(-n^5 - 5n^4 - 5n^3 + 5n^2 + 2n - 8)(3n+2)^{i-j}(4n+2)^j(i-j+1) \\
& + (2/9)(-2n^4 - 2n^3 + 4n^2)(3n+4)^{i-j}(4n+2)^j(i-j+1) \qquad (171) \\
\le & (3n+2)^{i-j}(4n+2)^j(i-j+1)(-(18/121)n^5) \\
& + (3n+4)^{i-j}(4n+2)^j(i-j+1)(-(4/9)n^4) \qquad (172)
\end{aligned}$$

For $\widehat{B}_{i,j} + \widetilde{B}_{i,j}$, we have (since $n \ge 3$)

$$\begin{aligned}
\widehat{B}_{i,j}(n) + \widetilde{B}_{i,j}(n) \le & (3n+2)^{i-j}(3n+1)^j(i-j+1)((41/9)n^5) \qquad (173) \\
& + (3n+2)^{i-j+1}(3n+1)^{j-1}j((17/9)n^5) \qquad (174) \\
& + (3n+2)^{i-j}(3n+3)^j(i-j+1)((48/9)n^5) \qquad (175) \\
& + (3n+2)^{i-j+1}(3n+3)^{j-1}j((2/3)n^5) \qquad (176) \\
& + (3n+4)^{i-j}(3n+1)^j(i-j+1)((14/3)n^4) \qquad (177) \\
& + (3n+4)^{i-j+1}(3n+1)^{j-1}j((38/9)n^4) \qquad (178) \\
& + (3n+4)^{i-j}(3n+3)^j(i-j+1)((16/3)n^4) \qquad (179) \\
& + (3n+4)^{i-j+1}(3n+3)^{j-1}j((4/3)n^4). \qquad (180)
\end{aligned}$$

We obtain $\widehat{B}_{i,j}(n) + \widetilde{B}_{i,j}(n) + \widehat{C}_{i,j} \le 0$ if both

$$\frac{18}{121} \ge \frac{41}{9}\left(\frac{3n+1}{4n+2}\right)^j + \frac{17}{9}\frac{3n+2}{4n+2}\left(\frac{3n+1}{4n+2}\right)^{j-1}\frac{j}{3} + \frac{48}{9}\left(\frac{3n+3}{4n+2}\right)^j + \frac{2}{3}\frac{3n+2}{4n+2}\left(\frac{3n+3}{4n+2}\right)^{j-1}\frac{j}{3} \qquad (181)$$

and

$$\frac{4}{9} \ge \frac{14}{3}\left(\frac{3n+1}{4n+2}\right)^j + \frac{38}{9}\frac{3n+4}{4n+2}\left(\frac{3n+1}{4n+2}\right)^{j-1}\frac{j}{3} + \frac{16}{3}\left(\frac{3n+3}{4n+2}\right)^j + \frac{4}{3}\frac{3n+4}{4n+2}\left(\frac{3n+3}{4n+2}\right)^{j-1}\frac{j}{3}. \qquad (182)$$

Since $n \ge 3$, it is sufficient to show that

$$\begin{aligned}
\frac{18}{121} \ge & \frac{41}{9}\left(\frac{3}{4}\right)^j + \frac{17}{9}\frac{11}{14}\left(\frac{3}{4}\right)^{j-1}\frac{j}{3} + \frac{48}{9}\left(\frac{12}{14}\right)^j + \frac{2}{3}\frac{11}{14}\left(\frac{12}{14}\right)^{j-1}\frac{j}{3} \qquad (183) \\
\frac{4}{9} \ge & \frac{14}{3}\left(\frac{3}{4}\right)^j + \frac{38}{9}\frac{13}{14}\left(\frac{3}{4}\right)^{j-1}\frac{j}{3} + \frac{16}{3}\left(\frac{12}{14}\right)^j + \frac{4}{3}\frac{13}{14}\left(\frac{12}{14}\right)^{j-1}\frac{j}{3}. \qquad (184)
\end{aligned}$$

Both of the above inequalities 183 and 184 hold for $j \ge 29$. □

Proof of Lemma 16. Because of Auxiliary Lemmas 19 and 20, it is sufficient to show that

$$\widetilde{B}_{i,j}(n) + \widetilde{D}_{i,j}(n) \le 0 \qquad (185)$$

whenever the conditions in Lemma 16 are satisfied. This inequality holds when

$$\begin{aligned}
&\underbrace{(4n+4)^{i-j+1}(-2n^3-2n^2+2n+2)((3n+1)^{j-1}-2(2n+2)^{j-1}+(n+3)^{j-1})}_{=:E_1}\\
&\quad+\underbrace{(4n+4)^{i-j+1}(-4n^2+8n-4)((3n+3)^{j-1}-2(2n+2)^{j-1}+(n+1)^{j-1})}_{=:E_2}\\
&\quad+\underbrace{(3n+2)^{i-j+1}(3n+1)^{j-1}(n^5+2n^4+2n^3-3n-2)}_{=:E_3}\\
&\quad+\underbrace{(3n+2)^{i-j+1}(3n+3)^{j-1}(2n^4-2n^3+2n^2-6n+4)}_{=:E_4}\\
&\quad+\underbrace{(3n+4)^{i-j+1}(3n+1)^{j-1}(2n^4+6n^3+2n^2-6n-4)}_{=:E_5}\\
&\quad+\underbrace{(3n+4)^{i-j+1}(3n+3)^{j-1}(4n^3-12n+8)}_{=:E_6}\leq 0.
\end{aligned} \tag{186}$$

The sufficient inequalities $E_1+E_3+E_5\leq 0$ and $E_2+E_4+E_6\leq 0$ hold if both of the following inequalities hold:

$$0\geq -4/25+\left(\frac{3}{4}\right)^{i-j+1}(n^2+2n+2)+\left(\frac{13}{16}\right)^{i-j+1}(2n+8/3) \tag{187}$$

$$0\geq -8/27+\left(\frac{3}{4}\right)^{i-j+1}2n^2+\left(\frac{13}{16}\right)^{i-j+1}4n \tag{188}$$

Here, we used that $j\geq 3$ and $n\geq 3$. Inequalities 187 and 188 hold for $n=35$ and $i-j+1=34$. Therefore they also hold for $3\leq n\leq 35$ and $i-j+1\geq 34$, establishing the desired lemma. □

Proof of Lemma 17. We define

$$EE:=(2n^4jp_8(n)^{i-j+1}+n^5jp_7(n)^{i-j+1})(-2\widetilde{p}_5(n)^{j-1}+\widetilde{p}_3(n)^{j-1}) \tag{189}$$

and then observe that

$$\mathrm{Pow}_{i,j}(n)=\widehat{A}_{i,j}(n)+\widehat{B}_{i,j}(n)+\widehat{C}_{i,j}(n)+\underbrace{\widetilde{A}_{i,j}(n)-EE}_{=:AA}+\underbrace{\widetilde{B}_{i,j}(n)+EE}_{=:BB}+\widetilde{D}_{i,j}(n). \tag{190}$$

By Auxiliary Lemma 19, we know that $\widetilde{D}_{i,j}(n)\leq 0$. Furthermore we obtain $AA\leq 0$ by the method used in the proof of Auxiliary Lemma 19 (proof that $\widetilde{A}_{i,j}(n)\leq 0$). In the following, we show that $\widehat{A}_{i,j}(n)+\widehat{B}_{i,j}(n)+\widehat{C}_{i,j}(n)+BB\leq 0$. For the first three terms, we estimate

$$\widehat{A}_{i,j}(n)+\widehat{B}_{i,j}(n)+\widehat{C}_{i,j}(n)\leq (i-j+1)(P_2Q_2+P_5Q_5) \tag{191}$$

by the proof of Auxiliary Lemma 20. Using $n\geq 36$ and $i-j\geq 3j\geq 9$, we continue with

$$P_2\leq -1.99n^4 \qquad Q_2\geq 0.94p_8(n)^{i-j}(\widetilde{p}_9(n)^j-2\widetilde{p}_8(n)^j-\widetilde{p}_7(n)^j+\widetilde{p}_6(n)^j+2\widetilde{p}_5(n)^j-\widetilde{p}_3(n)^j) \tag{192}$$

$$P_5\leq -n^5 \qquad Q_5\geq 0.94p_7(n)^{i-j}(\widetilde{p}_9(n)^j-2\widetilde{p}_8(n)^j-\widetilde{p}_7(n)^j+\widetilde{p}_6(n)^j+2\widetilde{p}_5(n)^j-\widetilde{p}_3(n)^j). \tag{193}$$

For BB, we estimate (using $n \geq 36$):

$$BB \leq jp_7(n)^{i-j+1}\Big(n^5(\widetilde{p}_7(n)^{j-1} - 2\widetilde{p}_5(n)^{j-1} + \widetilde{p}_3(n)^{j-1}) + 4.01n^4\widetilde{p}_8(n)^{j-1}\Big) \tag{194}$$

$$+ jp_8(n)^{i-j+1}\Big(2n^4(\widetilde{p}_7(n)^{j-1} - 2\widetilde{p}_5(n)^{j-1} + \widetilde{p}_3(n)^{j-1}) + 10.1n^3\widetilde{p}_8(n)^{j-1}\Big) \tag{195}$$

$$\leq 1.01jp_7(n)^{i-j}\Big(n^5(\widetilde{p}_7(n)^{j} - 2\widetilde{p}_5(n)^{j} + \widetilde{p}_3(n)^{j}) + 4.01n^4\widetilde{p}_8(n)^{j}\Big) \tag{196}$$

$$+ 1.03jp_8(n)^{i-j}\Big(2n^4(\widetilde{p}_7(n)^{j} - 2\widetilde{p}_5(n)^{j} + \widetilde{p}_3(n)^{j}) + 10.1n^3\widetilde{p}_8(n)^{j}\Big). \tag{197}$$

Using $i \geq 4j$ we have $i - j + 1 \geq 3j$ and thus obtain that $\widehat{A}_{i,j}(n) + \widehat{B}_{i,j}(n) + \widehat{C}_{i,j}(n) + BB \leq 0$ if both of the following inequalities are fulfilled:

$$\begin{aligned} &n^5(3 * 0.94(\widetilde{p}_9(n)^j - 2\widetilde{p}_8(n)^j - \widetilde{p}_7(n)^j + \widetilde{p}_6(n)^j + 2\widetilde{p}_5(n)^j - \widetilde{p}_3(n)^j) \\ &\qquad - 1.01(\widetilde{p}_7(n)^j - 2\widetilde{p}_5(n)^j + \widetilde{p}_3(n)^j)) \geq 1.01 * 4.01n^4\widetilde{p}_8(n)^j \end{aligned} \tag{198}$$

$$\begin{aligned} &n^4(3 * 1.99 * 0.94(\widetilde{p}_9(n)^j - 2\widetilde{p}_8(n)^j - \widetilde{p}_7(n)^j + \widetilde{p}_6(n)^j + 2\widetilde{p}_5(n)^j - \widetilde{p}_3(n)^j) \\ &\qquad - 1.03 * 2(\widetilde{p}_7(n)^j - 2\widetilde{p}_5(n)^j + \widetilde{p}_3(n)^j)) \geq 1.03 * 10.1n^3\widetilde{p}_8(n)^j \end{aligned} \tag{199}$$

The above inequalities hold for $j = 3$ and $n = 36$ and thus also for all $j \geq 3$ and $n \geq 36$. □

Proof of Lemma 18. There are only a finite number of cases for i and j that fulfill the conditions in the lemma. We investigated all of these cases individually and found in all cases that $\mathrm{Pow}_{i,j}(n)$ is nonpositive for all n. Two examples of $\mathrm{Pow}_{i,j}(n)$ are given below; a full list is available from the authors upon request. In all cases of i and j, $\mathrm{Pow}_{i,j}(n)$ could be decomposed into one negative factor and a number of factors which are nonnegative for all n.

$\mathrm{Pow}_{7,4}(n) = -192(n-1)^2 n^2 (1+n)^4 (44 + 100n + 93n^2 + 39n^3 + 6n^4)$

$\mathrm{Pow}_{46,15}(n) = -8(n-1)^2 n^2 (1+n)^3 (5577378013695597741036955238 4+$
$1850439678572421195660445876224n + 3007785581167471581748824480 1536n^2+$
$3196879666780171709910776083906 56n^3 + 2500655891772240491538557693853696n^4+$
$1535509202907574127213155569998 2336n^5 + 7707117387117979996586788836605 9520n^6+$
$3250331442748456519514662756281 22112n^7 + 1174732220393175424809067059282444288n^8+$
$3692439953418266608495553512575336448n^9 + 10207846013403227632394854152984854528n^{10}+$
$25038300982670870344758691457956577280n^{11} + 54869270060048371328828738931833962496n^{12}+$
$108018736442113512759726788664023318528n^{13} + 191878895587888899540992538246101467136n^{14}+$
$308632211888103540098352546840251793408n^{15} + 450767609607563251695188366660832264192n^{16}+$
$599111039674357298687021450391894884352n^{17} + 725809328478901104240863190621173612544n^{18}+$
$802444867413763805886645384321776664576n^{19} + 810255459024663986122734172767241175040n^{20}+$
$747505097788924725379552056201174589440n^{21} + 630098041087248209342703754296268999680n^{22}+$
$485145432298484099881671471259113239040n^{23} + 340980025511046902148788732489370548736n^{24}+$
$218553759476953003998385127447798082048n^{25} + 127582922717354697315571139498389566336n^{26}+$
$67719184103853602611937621007687117888n^{27} + 32616140623674928032775579955575195776n^{28}+$
$14219734032541043490216540741684065568n^{29} + 5595221169295229634866152249388359130n^{30}+$
$1980108553966131147117963568083777903n^{31} + 627582679136361072213708826817205869n^{32}+$
$177220511456901270015144963225818826n^{33} + 44300688721660092522416482129433948n^{34}+$
$9722756241094846198580685519777917n^{35} + 1853725882574217240202186568398451n^{36}+$
$302845578118349075936079079028252n^{37} + 41661185496339221938466283532106n^{38}+$
$4725253901229481698549143477177n^{39} + 432310193496890214181987715987n^{40}+$
$31477194477232889999228080602n^{41} + 1840110606779257774111111008n^{42}+$
$86063341710909155343040059n^{43} + 2530252898781282531716013n^{44})$

$\square$

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